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The Weierstrass semigroups on the quotient curve of a plane curve of degree $\leqq 7$ by an involution

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ABSTRACT

First we describe the Weierstrass semigroups on a plane curve of degree $\leqq 6$. Using this description we determine the Weierstrass semigroups at a ramification point and a branch point on a double covering from a plane curve of degree $\leqq 6$. In the case of a double covering from a plane curve of degree 7 we determine all the Weierstrass semigroups at branch points.

1. Introduction

Let C be a smooth irreducible curve where a *curve* means a projective curve over an algebraically closed field k of characteristic 0. We denote by $k(C)$ the field of rational functions on C. For a point P of C we define the Weierstrass semigroup $H(P)$ at P as follows:

$$H(P) = \left\{ n \in \mathbb{N}_0 \;\middle|\; \exists f \in k(C) \text{ such that } (f)_\infty = nP \right\},$$

where $\mathbb{N}_0$ is the additive semigroup of nonnegative integers. Then $H(P)$ is a *numerical semigroup of genus* g, which means a subsemigroup of $\mathbb{N}_0$ whose complement is a finite set with cardinality g. When we set $\mathbb{N}_0 \backslash H(P) = \{i_1 < i_2 < \cdots < i_g\}$, the sequence $i_1, i_2, \ldots, i_g$ of the integers is called the *Weierstrass gap sequence* at P. For two plane curves C_1 and C_2 we denote by $I(C_1 \cap C_2, Q)$ the intersection multiplicity of C_1 and C_2 at a point Q.

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In Section 2 we describe the Weierstrass semigroup $H(P)$ at a point P on a smooth irreducible plane curve C of degree $d = 4, 5, 6$. We set

$$i_1 = \max\{I(C \cap l, P) \mid l \text{ is a line in } \mathbb{P}^2\},$$
$$i_2 = \max\{I(C \cap \Gamma, P) \mid \Gamma \text{ is a conic in } \mathbb{P}^2\},$$
$$i_3 = \max\{I(C \cap \Gamma, P) \mid \Gamma \text{ is a cubic in } \mathbb{P}^2\}.$$

In the case $d = 4$ the value i_1 determines the semigroup $H(P)$. In the case $d = 5$ the values i_1 and i_2 determine the semigroup $H(P)$. In the case $d = 6$ we need the values i_1, i_2, i_3 and i'_3 to determine $H(P)$ where

$$i'_3 = \max\{I(C \cap \Gamma, P) \neq i_3 \mid \Gamma \text{ is a cubic in } \mathbb{P}^2\}.$$

In order to show the existence of such a pointed curve (C, P) we give a polynomial defining the curve C with its point P as the origin. In the case $i_1 \geqq d - 3$ Coppens and Kato [2] described the Weierstrass semigroups $H(P)$ and showed the existence of a pointed smooth irreducible plane curve (C, P) with such a semigroup. But they did not give an explicit polynomial defining the curve C.

In Sections 3 and 4 we study the Weierstrass semigroups on a smooth irreducible plane curve C of degree d with an involution ι and the quotient curve $C_0 = C/\langle\iota\rangle$ of C by ι. Let $\pi : C \longrightarrow C_0 = C/\langle\iota\rangle$ be the covering map. In the case $d = 4, 5, 6$ using the results of Section 2 we obtain the Weierstrass semigroups at ramification points of π in Section 3. Kikuchi [3] gave the Weierstrass semigroups at fixed points by an involution ι on a smooth irreducible plane curve of degree 5 or 6. His method is different from ours. He determined them calculating the order sequence of regular 1-forms at the fixed point under the involution. In this paper we give a polynomial defining the curve C of degree d and calculate the intersection multiplicities of C with the curves of degree $d - 3$ at the origin which is the fixed point under the involution. In the case $d = 7$ we determine the Weierstrass semigroups at branch points of π in Section 4.

2. The Weierstrass semigroups on a plane curve of degree $\leqq 6$

Let C be a smooth irreducible plane curve of degree $d \geqq 2$. Let P be a point on C. We divide the lines on the plane into three types according to the intersection multiplicity with C at P:

$$\begin{cases} I(C \cap l_0, P) = 0, \\ I(C \cap l_1, P) = 1, \\ I(C \cap l_2, P) \geqq 2. \end{cases}$$

We call l_2 the tangent line to C at P and denote it by $T_P C$.

The following are well known:

Lemma 2.1.

(1) *On a smooth irreducible plane curve of degree $d \geqq 4$, the canonical series is cut out by the system of all curves of degree $d - 3$.*
(2) (*Namba's lemma*) *Let C_1, C_2 and C be three plane curves defined by homogeneous polynomials F_1, F_2 and F, respectively. Let P be a smooth point on C. If $I(C \cap C_1, P) \geqq m$ and $I(C \cap C_2, P) \geqq m$, then $I(C_1 \cap C_2, P) \geqq m$.*

We will use the following lemma frequently when we prove the nonsingularity of a given curve using Bertini's theorem.

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Lemma 2.2. *Let C be a plane curve defined by an irreducible homogeneous polynomial $F(x, y, z)$ of degree $d \geqq 2$. If F has a term ax^d or $ax^{d-1}y$ or $ax^{d-1}z$ with $a \neq 0$ then the point $(1, 0, 0)$ is not a singular point of C. Also we have similar statements with respect to the points $(0, 1, 0)$ and $(0, 0, 1)$.*

Proof. It follows from easy computation. □

2.1. The case $d = 4$

Let C be a smooth irreducible plane curve of degree 4. The genus $g(C)$ of C is equal to 3 and the canonical series is cut out by the system of all lines. We have $2 \leqq i_1 \leqq 4$. Then the order sequence of canonical series at P is $\{0, 1, i_1\}$, or equivalently, the gap sequence at P is $\{1, 2, i_1 + 1\}$, and hence the Weierstrass semigroup at P is $\{0, 3 \to i_1, i_1 + 2 \to\}$, where "$\to$" denotes the consecutive integers. If $i_1 = 2$, then P is not a Weierstrass point and we get such Weierstrass semigroup at a general point on C. If $3 \leqq i_1 \leqq 4$, then P is a Weierstrass point. Now we give an example for each case.

(I) $i_1 = 4$.

Let C be a curve defined by the polynomial $yz^3 - x^4 + y^4$, and let $P = (0, 0, 1) \in C$. Then we can prove that C is smooth and $i_1 = 4$.

(II) $i_1 = 3$.

Let C be a curve defined by the polynomial $yz^3 - x^3z + y^4$, and let $P = (0, 0, 1) \in C$. Then we can prove that C is smooth and $i_1 = 3$.

Thus the list of all Weierstrass gap sequences on a smooth plane curve of degree 4 are $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 2, 5\}$.

2.2. The case $d = 5$

Let C be a smooth irreducible plane curve of degree 5. The genus $g(C)$ of C is equal to 6 and the canonical series is cut out by the system of all conics. Then $2 \leqq i_1 \leqq 5$, and $5 \leqq i_2 \leqq 10$, since the dimension of the system of conics is 5. Let f_2 be a conic such that $I(C \cap f_2, P) = i_2$. We consider the cases one by one.

(I) $i_1 = 5$.

By Bézout's theorem, we have $i_2 = 10$. Since the canonical series is cut out by conics, we obtain the order sequence of canonical series at P is $\{0, 1, 2, 5, 6, 10\}$, or equivalently, the gap sequence at P is $\{1, 2, 3, 6, 7, 11\}$, and hence the Weierstrass semigroup at P is $\{0, 4, 5, 8 \to 10, 12 \to\}$. Using elementary computation, we can prove that the curve C defined by a polynomial $f := yz^4 - x^5 + y^5$, and a point $P = (0, 0, 1)$ on C satisfy such a condition $i_1 = 5$.

(II) $i_1 = 4$.

We have $i_2 = 8$. We obtain the order sequence of canonical series at P is $\{0, 1, 2, 4, 5, 8\}$, or equivalently, the gap sequence at P is $\{1, 2, 3, 5, 6, 9\}$, and hence the Weierstrass semigroup at P is $\{0, 4, 7, 8, 10 \to\}$. Using elementary computation, we can prove that the curve C defined by a polynomial $f := yz^4 - x^4z + y^5$, and a point $P = (0, 0, 1)$ on C satisfy such a condition $i_1 = 4$.

(III) $i_1 = 3$.

We have $i_2 = 6$. We obtain the order sequence of canonical series at P is $\{0, 1, 2, 3, 4, 6\}$, or equivalently, the gap sequence at P is $\{1, 2, 3, 4, 5, 7\}$, and hence the Weierstrass semigroup at P is

$\{0, 6, 8 \rightarrow\}$. For existence, consider a linear system of curves with degree 5

$$\{f_{abc} := a(yz^2 - x^3)z^2 + b(yz^4 + x^5) + cy^5\}.$$

Then the base locus of the linear system is $\{(0,0,1)\}$ which is not singular point of f_{abc} if $a + b \neq 0$. Thus, by Bertini's theorem, for general, especially nonzero a, b, c, f_{abc} defines a smooth plane curve. Easily we can compute that $i_1 = I(y \cap f_{abc}, P) = 3$.

(IV) $i_1 = 2$.

The order sequence obtained by two lines, i.e., reducible conics is $0, 1, 2, 3, 4$. Since $i_2 \geqq 5$, we note it is cut out by an irreducible conic f_2. Thus the order sequence is $\{0, 1, 2, 3, 4, i_2\}$, or equivalently, the gap sequence at P is $\{1, 2, 3, 4, 5, i_2 + 1\}$, and hence the Weierstrass semigroup at P is $\{0, 6 \rightarrow i_2, i_2 + 2 \rightarrow\}$. If $i_2 = 5$, then P is not a Weierstrass point, for which we take a general point on C as an example of such point P. Now we prove the existence of smooth plane curves of degree 5 with $i_1 = 2$ and $6 \leqq i_2 \leqq 10$ by giving the defining homogeneous equation explicitly. Consider the linear system of polynomials with the form:

$$\{f_{abcd} := a(yz - x^2)z^3 + b(yz - x^2)(x^3 + y^3) + cx^j y^k z^{5-j-k} + dy^5\},$$

where $0 \leqq j \leqq 1$, $3 \leqq k \leqq 5$, and $j + k \leqq 5$. Then the base locus of the linear system is contained in $\{(1,0,0), (0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Since the tangent line at $P = (0,0,1)$ is $y = 0$, we compute $i_1 = I(y \cap f_{abcd}, P) = 2$ easily. Also, if we let $f_2 = yz - x^2$, then $i_2 = I(f_2 \cap f_{abcd}, P) = I(f_2 \cap (cx^j y^k z^{5-j-k} + dy^5), P) = 2k + j$. Thus we have a smooth curve with each $i_2 \in \{6, 7, 8, 9, 10\}$ by varying values j and k.

Summing up all cases in the above, we determined the list of all Weierstrass gap sequences on a smooth plane curve of degree 5 as follows.

1, 2, 3, 6, 7, 11 ($i_1 = 5$).
1, 2, 3, 5, 6, 9 ($i_1 = 4$).
1, 2, 3, 4, 5, 7 ($i_1 = 3$).
1, 2, 3, 4, 5, 11 ($i_1 = 2$, $i_2 = 10$).
1, 2, 3, 4, 5, 10 ($i_1 = 2$, $i_2 = 9$).
1, 2, 3, 4, 5, 9 ($i_1 = 2$, $i_2 = 8$).
1, 2, 3, 4, 5, 8 ($i_1 = 2$, $i_2 = 7$).
1, 2, 3, 4, 5, 7 ($i_1 = 2$, $i_2 = 6$).
1, 2, 3, 4, 5, 6 ($i_1 = 2$, $i_2 = 5$).

2.3. The case $d = 6$

Let C be a smooth irreducible plane curve of degree 6. The genus $g(C)$ of C is equal to 10 and the canonical series is cut out by the system of all cubics. Then $2 \leqq i_1 \leqq 6$, $5 \leqq i_2 \leqq 12$ and $9 \leqq i_3 \leqq 18$ since the dimensions of the system of conics and cubics is 5 and 9, respectively. Let f_1, f_2 and f_3 be a line, a conic and a cubic such that $I(C \cap f_j, P) = i_j$, $j = 1, 2, 3$. We consider the cases one by one.

(I) $i_1 = 6$.

By Bézout's theorem, we have $i_2 = 12$ and $i_3 = 18$. Since the canonical series is cut out by cubics, we obtain the order sequence of canonical series at P is $\{0 \rightarrow 3, 6 \rightarrow 8, 12, 13, 18\}$, which is obtained by using only three lines, i.e., reducible cubics. Thus the gap sequence at P is $\{1 \rightarrow 4, 7 \rightarrow 9, 13, 14, 19\}$, and hence the Weierstrass semigroup at P is $\{0, 5, 6, 10, 11, 12, 15 \rightarrow 18, 20 \rightarrow\}$. Using elementary

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computation, we can prove that the curve C defined by a polynomial $f := yz^5 - x^6 + y^6$, and a point $P = (0, 0, 1)$ on C satisfy such a condition $i_1 = 6$.

(II) $i_1 = 5$.

We have $i_2 = 10$ and $i_3 = 15$. Since the canonical series is cut out by cubics, we obtain the order sequence of canonical series at P is $\{0 \to 3, 5 \to 7, 10, 11, 15\}$, or equivalently, the gap sequence at P is $\{1 \to 4, 6 \to 8, 11, 12, 16\}$, and hence the Weierstrass semigroup at P is $\{0, 5, 9, 10, 13, 14, 15, 18 \to\}$. Using elementary computation, we can prove that the curve C defined by a polynomial $f := yz^5 - x^5z + y^6$, and a point $P = (0, 0, 1)$ on C satisfy such a condition $i_1 = 5$.

(III) $i_1 = 4$.

We have $i_2 = 8$ and $i_3 = 12$. Since the canonical series is cut out by cubics, we obtain the order sequence of canonical series at P is $\{0 \to 6, 8, 9, 12\}$, or equivalently, the gap sequence at P is $\{1 \to 7, 9, 10, 13\}$, and hence the Weierstrass semigroup at P is $\{0, 8, 11, 12, 14 \to\}$. For existence, consider a linear system of curves with degree 6

$$\{f_{abc} := a(yz^3 - x^4)z^2 + b(yz^5 - x^6) + cy^6\}.$$

Then the base locus of the linear system is $\{(0, 0, 1)\}$ which is not singular point of f_{abc} if $a + b \neq 0$. Thus, by Bertini's theorem, for general, especially nonzero a, b, c, f_{abc} defines a smooth plane curve. Easily we can compute that $i_1 = I(y \cap f_{abc}, P) = 4$.

(IV) $i_1 = 3$.

We have $i_2 = 6$. Since the dimension of the system of cubics is 9, we have $i_3 \geqq 9$. A part of the order sequence $\{0 \to 7, 9\}$ is obtained by reducible cubics consisting of three lines. Since the genus is 10, we need one more order which will be cut out by irreducible cubic. Here we divide into several cases according to the value i_3.

(1) $i_3 = 18$.

The order sequence is $\{0 \to 7, 9, 18\}$, or equivalently, the gap sequence is $\{1 \to 8, 10, 19\}$, and hence $H(P) = \{0, 9, 11 \to 18, 20 \to\}$.

(2) $i_3 = 17$.

The order sequence is $\{0 \to 7, 9, 17\}$, or equivalently, the gap sequence is $\{1 \to 8, 10, 18\}$, and hence $H(P) = \{0, 9, 11 \to 17, 19 \to\}$, which is a contradiction since if $9 \in H(P)$ then $18 \in H(P)$. Thus this case cannot occur.

(3) $10 \leqq i_3 \leqq 16$.

The order sequence is $\{0 \to 7, 9, i_3\}$, the gap sequence $\{1 \to 8, 10, i_3 + 1\}$, and $H(P) = \{0, 9, 11 \to i_3, i_3 + 2 \to\}$.

(4) $i_3 = 9$.

Then there exists another cubic f_3' such that $I(C \cap f_3', P) = 8$, hence the order sequence is $\{0 \to 9\}$ and the gap sequence is $\{1 \to 10\}$. Thus $H(P) = \{0, 11 \to\}$, and P is not a Weierstrass point.

Now we prove the existence of such smooth plane curves. Consider the linear system of polynomials with the form:

$$\{f_{abcd} := a(yz^2 - x^3)z^3 + b(yz^2 - x^3)(x^3 + y^3) + cx^jy^kz^{6-j-k} + dy^6\},$$

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where $0 \leqq j \leqq 2$, $3 \leqq k \leqq 6$, and $j + k \leqq 6$. Then the base locus of the linear system is contained in $\{(1,0,0),(0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Since the tangent line at $P = (0,0,1)$ is $y = 0$, we compute $i_1 = I(y \cap f_{abcd}, P) = 3$ easily. Also, if we let $f_3 = yz^2 - x^3$, then $i_3 = I(f_3 \cap f_{abcd}, P) = I(f_3 \cap (cx^j y^k z^{6-j-k} + dy^6), P) = 3k + j$. Thus we have a smooth curve with each $i_3 \in \{10, 11, 12, 13, 14, 15, 16, 18\}$ by varying values j and k.

(V) $i_1 = 2$.

Then i_2 $(\geqq 5)$ is attained by an irreducible conic. The part of the order sequence $\{0, 1, 2, 3, 4, 5, 6\}$ is cut out by reducible cubics consisting of three lines, and we need three more numbers to determine the order sequence. We divided into several cases according to the value of i_2.

(1) $i_1 = 2$ and $i_2 \geqq 7$.

Then we have $i_3 = i_2 + 2$ by Bézout's theorem. The order sequence is determined as $\{0 \to 6, i_2, i_2 + 1, i_2 + 2\}$, the gap sequence is $\{1 \to 7, i_2 + 1, i_2 + 2, i_2 + 3\}$, and $H(P) = \{0, 8 \to i_2, i_2 + 4 \to\}$. Note that if $i_2 = 7$, then P is not a Weierstrass point.

We prove the existence of such smooth plane curves. Consider the linear system of polynomials with the form:

$$\left\{f_{abcd} := a\left(yz - x^2\right)z^4 + b\left(yz - x^2\right)\left(x^4 + y^4\right) + cx^j y^k z^{6-j-k} + dy^6\right\},$$

where $0 \leqq j \leqq 1$, $4 \leqq k \leqq 6$, and $j + k \leqq 6$. Then the base locus of the linear system is contained in $\{(1,0,0),(0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Since the tangent line at $P = (0,0,1)$ is $y = 0$, we compute $i_1 = I(y \cap f_{abcd}, P) = 2$ easily. Also, if we let $f_2 = yz - x^2$, then $i_2 = I(f_2 \cap f_{abcd}, P) = I(f_2 \cap (cx^j y^k z^{6-j-k} + dy^6), P) = 2k + j$. Thus we have a smooth curve with each $i_2 \in \{8, 9, 10, 11, 12\}$ by varying values j and k.

(2) $i_1 = 2$ and $i_2 = 6$.

The orders obtained by reducible cubics consisting of lines and conics are $0 \to 6, 7, 8$, hence we need one more order, which is just i_3. Thus the order sequence is $\{0 \to 8, i_3\}$, $9 \leqq i_3 \leqq 18$, the gap sequence is $\{1 \to 9, i_3 + 1\}$, and $H(P) = \{0, 10 \to i_3, i_3 + 2 \to\}$. Note that if $i_3 = 9$, then P is not a Weierstrass point.

Now we prove the existence for each smooth plane curve of degree 6 with $i_1 = 2$, $i_2 = 6$ and $i_3 \in \{10 \to 16, 18\}$. Consider the linear system of polynomials with the form:

$$\left\{f_{abcd} := a\left(yz^2 - x^2z - y^3\right)z^3 + b\left(yz^2 - x^2z - y^3\right)x^3 + c\left(yz^2 - x^2z - y^3\right)y^3 \right.$$
$$\left. + dx^j y^k \left(yz - x^2\right)^l z^{6-j-k-2l}\right\},$$

where $0 \leqq j \leqq 1$, $0 \leqq k \leqq 2$, $1 \leqq l \leqq 3$, and $j + k + 2l \leqq 6$. Then the base locus of the linear system is contained in

$$\left\{(1,0,0),(0,1,0),(0,0,1),(0,1,1),(0,-1,1)\right\},$$

each of which is not a singular point by Lemma 2.2. Indeed, since $j \leqq 1$, $\frac{\partial f_{abcd}}{\partial x}(0, \pm 1, 1) \neq 0$ if $d \neq 0$. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Since the tangent line at $P = (0,0,1)$ is $y = 0$, we compute $i_1 = I(y \cap f_{abcd}, P) = 2$ easily. Also, if we let $f_2 = yz - x^2$, then

$$i_2 = I(f_2 \cap f_{abcd}, P) = I\big(f_2 \cap y^3\big(-az^3 - bx^3 - cy^3\big), P\big)$$
$$= I\big(f_2 \cap y^3, P\big) = 6.$$

If we let $f_3 = yz^2 - x^2z - y^3$, then

$$i_3 = I(f_3 \cap f_{abcd}, P)$$
$$= I\big(f_3 \cap \big(dx^j y^k \big(yz - x^2\big)^l z^{6-j-k-2l}\big), P\big) = j + 2k + 6l.$$

Thus we have a smooth curve with each $i_3 \in \{10 \to 16, 18\}$ by varying values j, k and l.

In the above construction, we do not have the example with $i_1 = 2$, $i_2 = 6$ and $i_3 = 17$. In fact, we can prove that such a curve does not exist as follows.

Theorem 2.3. *There does not exist a smooth plane curve of degree 6 with $i_1 = 2$, $i_2 = 6$ and $i_3 = 17$.*

Proof. Suppose that such a smooth plane sextic C_6 exists. Let C_2 [resp. C_3] be a conic [resp. a cubic] satisfying that $I(C_6 \cap C_2, P) = 6$ [resp. $I(C_6 \cap C_3, P) = 17$].

Case. P is a smooth point on C_3.

Consider two divisors $C_3 . C_2^3 = 18P$ and $C_3 . C_6 = 17P + Q$ on C_3, which are linearly equivalent. Thus $P \sim Q$. We prove that $P = Q$. Let L be any line through P. Since $L . C_3 \sim L . C_3 - P + Q$ and the linear series on the plane curve cut out by the system of curves of the same degree is complete, the divisor $L . C_3 - P + Q$ is cut out by a line, say L'. Since L contains two more points except P, L' should be equal to L. Thus any line through P contains Q, which implies $P = Q$.

Case. P is not a smooth point on C_3.

Then P is a node or a cusp of C_3. If P is a cusp, then we can prove that $I(C_3 \cap C_6, P) = 3$ using a parametrization, which is a contradiction. Thus P should be a node. Then there are two branches of C_3 centered at P, say B_1 and B_2. We may assume B_1 is tangent to C_6 at P. Then $I(B_1 \cap C_6, P) = 16$. By Namba's lemma, we have $I(B_1 \cap C_2, P) = 6$. Then $I(C_3 \cap C_2, P) = 7$ which is a contradiction. □

(3) $i_1 = 2$ and $i_2 = 5$ ($i_3 \geqq 9$).

In this case we have $i_3 \geqq 9$ which is cut out by an irreducible cubic. The orders obtained by reducible cubics consisting of lines and conics are $0 \to 7$, hence we need two more orders $\geqq 8$. One is just i_3, and the other one, say i_3', is cut out by an irreducible cubic f_3'. By definition of i_3, we have $8 \leqq i_3' < i_3$. Then, by Bézout's theorem, we have $i_3' = 8$ or 9.

(3-1) $i_1 = 2$, $i_2 = 5$, $i_3' = 9$ and $10 \leqq i_3 \leqq 18$.

The order sequence is $\{0 \to 7, 9, i_3\}$, the gap sequence is $\{1 \to 8, 10, i_3 + 1\}$, and $H(P) = \{0, 9, 11 \to i_3, i_3 + 2 \to\}$.

Theorem 2.4. *There exists a smooth plane curve with $i_1 = 2$, $i_2 = 5$, $i_3' = 9$ and $i_3 \in \{10 \to 16, 18\}$. There does not exists a smooth plane curve with $i_1 = 2$, $i_2 = 5$, $i_3' = 9$ and $i_3 = 17$.*

Proof. If $i_3 = 17$, then 18 is a gap at P, which is impossible since $9 \in H(P)$.

Now we construct curves with $i_3 \neq 17$. Let $f_1 = y$, $f_2 = yz - x^2$, $f_3 = yz^2 - x^2z + xy^2$, and $f_3' = xyz - x^3 + y^3$. Around the point $P = (0, 0, 1)$, letting $z = 1$, since

$$xy - x^3 + y^3 = x\big(y - x^2 + xy^2\big) - x^2y^2 + y^3 = x\big(y - x^2 + xy^2\big) + y^2\big(y - x^2\big),$$

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we have

$$I\big(f_3\cap f_3', P\big) = I\big(f_3\cap y^2\big(y-x^2\big), P\big) = 2\cdot 2+5 = 9.$$

1) Case $i_3 = 18$.

Consider the linear system of polynomials with the form:

$$\big\{f_{abcd} := af_3z^3 + bf_3x^3 + cf_3y^3 + d\big(f_3'\big)^2\big\}.$$

Then the base locus of the linear system is contained in $\{(1,0,0),(0,1,0),(0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Also we compute the numbers

$$i_1 = I(f_1\cap f_{abcd}, P) = 2, \qquad i_2 = I(f_2\cap f_{abcd}, P) = 5,$$
$$i_3 = I(f_3\cap f_{abcd}, P) = 2I(f_3\cap f_3', P) = 18, \qquad i_3' = I\big(f_3'\cap f_{abcd}, P\big) = I\big(f_3'\cap f_3, P\big) = 9.$$

2) Case $i_3 = 16$.

Consider the linear system of polynomials with the form:

$$\big\{f_{abcd} := af_3z^3 + bf_3x^3 + cf_3y^3 + df_3'f_2f_1\big\}.$$

Then the base locus of the linear system is contained in $\{(1,0,0),(0,1,0),(0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Also we compute the numbers

$$i_1 = I(f_1\cap f_{abcd}, P) = 2, \qquad i_2 = I(f_2\cap f_{abcd}, P) = 5,$$
$$i_3 = I(f_3\cap f_{abcd}, P) = I\big(f_3\cap f_3'f_2f_1, P\big) = 16, \qquad i_3' = I\big(f_3'\cap f_{abcd}, P\big) = I\big(f_3'\cap f_3, P\big) = 9.$$

3) Case $10 \leqq i_3 \leqq 15$.

Consider the linear system of polynomials with the form:

$$\big\{f_{abcd} := af_3z^3 + bf_3x^3 + cf_3y^3 + dx^jy^k\big(yz-x^2\big)^lz^{6-j-k-2l}\big\},$$

where $0 \leqq j \leqq 1$, $0 \leqq k \leqq 2$, $2 \leqq l \leqq 3$, and $j+k+2l \leqq 6$. Then the base locus of the linear system is contained in $\{(1,0,0),(0,1,0),(0,0,1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Also we compute the numbers

$$i_1 = I(f_1\cap f_{abcd}, P) = 2, \qquad i_2 = I(f_2\cap f_{abcd}, P) = 5,$$
$$i_3 = I(f_3\cap f_{abcd}, P) = j+2k+5l, \qquad i_3' = I\big(f_3'\cap f_{abcd}, P\big) = I\big(f_3'\cap f_3, P\big) = 9.$$

Thus we have a smooth curve with each $i_3 \in \{10 \to 15\}$ by varying values j, k and l. □

(3-2) $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ and $9 \leqq i_3 \leqq 18$.

The order sequence is $\{0 \to 8, i_3\}$, the gap sequence is $\{1 \to 9, i_3+1\}$, and $H(P) = \{0, 10 \to i_3, i_3+2 \to\}$. We prove that there exists a smooth plane curve with each $i_3 \in \{9 \to 18\}$ in the following theorem.

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Theorem 2.5. *There exists a smooth plane curve with* $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ *and* $i_3 \in \{10 \to 18\}$.

Proof. Note that if $i_3 = 9$, then P is not a Weierstrass point, hence the existence is trivial.

Now we construct curves with $10 \leqq i_3 \leqq 18$ case by case.

1) $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ and $10 \leqq i_3 \leqq 15$.

Let $f_1 = y$, $f_2 = yz - x^2$, $f_3 = yz^2 - x^2z + xy^2 + y^3$, and $f_3' = xy - y^2 - x^3 + x^2y + y^3$. Then, at the point P, letting $z = 1$, we have

$$f_3' = (x - y + y^2) f_3 + y^4 - xy^4 - y^5,$$

hence

$$I(f_3 \cap f_3', P) = I(f_3 \cap y^4(1 - x - y), P) = 4I(f_3 \cap y, P) = 8.$$

Consider the linear system of polynomials with the form:

$$\{ f_{abcd} := af_3z^3 + bf_3x^3 + cf_3y^3 + dx^jy^k(yz - x^2)^l z^{6-j-k-2l} \},$$

where $0 \leqq j \leqq 1$, $0 \leqq k \leqq 2$, $2 \leqq l \leqq 3$, and $j + k + 2l \leqq 6$. Then the base locus of the linear system is contained in $\{(1,0,0), (0,1,0), (0,0,1), (-1,1,1), (-1,1,0)$ (in case $6 - j - k - 2l$ is positive), $(0, 1, \pm\sqrt{-1})$ (in case j is positive)$\}$, each of which is not a singular point by Lemma 2.2. Indeed, since $(-1, 1, 1)$ is a nonsingular point of f_3, it is a nonsingular point of a general member of the linear system. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Also we compute the numbers

$$i_1 = I(f_1 \cap f_{abcd}, P) = 2, \qquad i_2 = I(f_2 \cap f_{abcd}, P) = 5,$$
$$i_3 = I(f_3 \cap f_{abcd}, P) = j + 2k + 5l, \qquad i_3' = I(f_3' \cap f_{abcd}, P) = I(f_3' \cap f_3, P) = 8.$$

Thus we have a smooth curve with each $i_3 \in \{10 \to 15\}$ by varying values j, k and l.

2) $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ and $i_3 = 16$.

Let f_1, f_2, f_3, and f_3' be as above. Consider the linear system of polynomials with the form:

$$\{ f_{abcd} := af_3z^3 + bf_3x^3 + cf_3y^3 + d(f_3')^2 \}.$$

Then the base locus of the linear system is contained in $\{(1,0,0), (0,1,0), (0,0,1), (\frac{2}{3}, \frac{1}{3}, 1)\}$, each of which is not a singular point by Lemma 2.2. By Bertini's theorem, for general, especially nonzero, a, b, c and d, the polynomial defines a smooth curve. Also we compute the numbers

$$i_1 = I(f_1 \cap f_{abcd}, P) = 2, \qquad i_2 = I(f_2 \cap f_{abcd}, P) = 5,$$
$$i_3 = I(f_3 \cap f_{abcd}, P) = 2I(f_3 \cap f_3', P) = 16, \qquad i_3' = I(f_3' \cap f_{abcd}, P) = I(f_3' \cap f_3, P) = 8.$$

3) $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ and $i_3 = 17$.

Let the point P and inhomogeneous polynomials f_2, f_3, f_3', f_6 as follows:

$$P = (0,0), \qquad f_2 = y - x^2 + y^2, \qquad f_3 = y - x^2 + x^2y + xy^2,$$
$$f_3' = y - x^2 - xy + y^2 + x^3,$$

$$f_6 = y - x^2 + x^2y + xy^2 + 2y^3 - 2x^2y^2 - 2xy^3 + 3x^3y^2 + xy^4 + 2y^5 \\ - x^5y + 2x^4y^2 - 5x^2y^4 - 2xy^5 + y^6.$$

Since $f_3 = (1-y)f_2 + y^2(y+x)$, we have $I(f_3 \cap f_2, P) = I(y^2(y+x) \cap f_2, P) = 5$. Also, using elementary computation, we have $f_3' = (1 - x + y - xy - xy^2)f_3 + x^2y^3(y+x+1)$, hence we obtain $I(f_3 \cap f_3', P) = 8$. Similarly we have $f_6 := q(x)f_3 - x^3y^7(x+y+1)^2$ where

$$q(x) = 1 + 2y^2 - 2xy^2 + x^3y - 2x^2y^2 - xy^3 + 2y^4 + x^3y^2 - x^2y^3 - 2xy^4 \\ + x^3y^3 + y^5 - 2xy^5 + x^3y^4 + x^2y^5 - xy^6 + x^3y^5 + 2x^2y^6 + x^3y^6 + x^2y^7.$$

Thus $I(f_3 \cap f_6, P) = 3I(f_3 \cap x, P) + 7I(f_3 \cap y, P) = 17$. Since we have the divisor $f_3 \cdot f_6 = 17P + Q$ for some point Q, and we conclude the point Q is a nonsingular point of both curves. Thus, using Bertini's theorem, for general nonzero a, b, c, d, $f_3(a + bx^3 + cy^3) + df_6$ defines a smooth curve C of degree 6, and we have $I(C \cap f_3, P) = 17$ and

$$I(C \cap f_2, P) = I(f_3 \cap f_2, P) = 5, \qquad I\big(C \cap f_3', P\big) = I\big(f_3 \cap f_3', P\big) = 8,$$

by Namba's lemma.

4) $i_1 = 2$, $i_2 = 5$, $i_3' = 8$ and $i_3 = 18$.

Let the point P and inhomogeneous polynomials f_2, f_3, f_3', f_6 as follows:

$$P = (0,0), \qquad f_2 = y - x^2 + xy + 2y^2, \qquad f_3 = y - x^2 + xy + y^2 + x^2y + xy^2, \\ f_3' = y - x^2 + 3y^2 + x^3 - 2x^2y + xy^2 + 2y^3, \\ f_6 = y - x^2 + 2y^2 + x^3 - x^2y - x^4y + 4x^3y^2 - 11x^2y^3 - xy^4 + 8y^5 \\ + x^6 - 7x^5y + 13x^4y^2 + 4x^3y^3 - 17x^2y^4 + 8y^6.$$

Since $f_3 = f_2(1-y) + 2y^2(y+x)$ and $f_3' = f_3(1 - x + 2y - xy - xy^2) + y^3x(x^2 + 2x + xy + y)$, we have

$$I(f_3 \cap f_2, P) = I\big(y^2(y+x) \cap f_2, P\big) = 5, \\ I\big(f_3 \cap f_3', P\big) = I\big(f_3 \cap xy^3\big(y + 2x + x^2 + xy\big), P\big) = 8.$$

Substituting the congruence equation $x^2 \equiv y(1 + x + y + x^2 + xy) \pmod{f_3}$ over and over into f_6, we have

$$f_6 \equiv \big(8 + 32x + 63x^2 + 66x^3 + 37x^4 + 10x^5 + x^6\big)y^9 \\ + \big(9 + 31x + 43x^2 + 29x^3 + 9x^4 + x^5\big)y^{10} \pmod{f_3}.$$

Thus $I(f_3 \cap f_6, P) = 9I(f_3 \cap y, P) = 18$. Using Bertini's theorem, for general nonzero a, b, c, d, $f_3(a + bx^3 + cy^3) + df_6$ defines a smooth curve C of degree 6, and we have $I(C \cap f_3, P) = 18$ and

$$I(C \cap f_2, P) = I(f_3 \cap f_2, P) = 5, \qquad I\big(C \cap f_3', P\big) = I\big(f_3 \cap f_3', P\big) = 8,$$

by Namba's lemma. □

Finally, summarizing the above, we give the list of all Weierstrass gap sequences on a smooth plane curve of degree 6.

1, 2, 3, 4, 7, 8, 9, 13, 14, 19 ($i_1 = 6$).
1, 2, 3, 4, 6, 7, 8, 11, 12, 16 ($i_1 = 5$).
1, 2, 3, 4, 5, 6, 7, 9, 10, 13 ($i_1 = 4$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 19 ($i_1 = 3$, $i_2 = 6$, $i_3 = 18$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 17 ($i_1 = 3$, $i_2 = 6$, $i_3 = 16$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 16 ($i_1 = 3$, $i_2 = 6$, $i_3 = 15$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 15 ($i_1 = 3$, $i_2 = 6$, $i_3 = 14$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 14 ($i_1 = 3$, $i_2 = 6$, $i_3 = 13$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 13 ($i_1 = 3$, $i_2 = 6$, $i_3 = 12$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 12 ($i_1 = 3$, $i_2 = 6$, $i_3 = 11$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 11 ($i_1 = 3$, $i_2 = 6$, $i_3 = 10$).
1, 2, 3, 4, 5, 6, 7, 13, 14, 15 ($i_1 = 2$, $i_2 = 12$).
1, 2, 3, 4, 5, 6, 7, 12, 13, 14 ($i_1 = 2$, $i_2 = 11$).
1, 2, 3, 4, 5, 6, 7, 11, 12, 13 ($i_1 = 2$, $i_2 = 10$).
1, 2, 3, 4, 5, 6, 7, 10, 11, 12 ($i_1 = 2$, $i_2 = 9$).
1, 2, 3, 4, 5, 6, 7, 9, 10, 11 ($i_1 = 2$, $i_2 = 8$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 19 ($i_1 = 2$, $i_2 = 6$, $i_3 = 18$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 17 ($i_1 = 2$, $i_2 = 6$, $i_3 = 16$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 16 ($i_1 = 2$, $i_2 = 6$, $i_3 = 15$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 15 ($i_1 = 2$, $i_2 = 6$, $i_3 = 14$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 14 ($i_1 = 2$, $i_2 = 6$, $i_3 = 13$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 13 ($i_1 = 2$, $i_2 = 6$, $i_3 = 12$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 12 ($i_1 = 2$, $i_2 = 6$, $i_3 = 11$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 11 ($i_1 = 2$, $i_2 = 6$, $i_3 = 10$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 19 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 18$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 17 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 16$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 16 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 15$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 15 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 14$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 14 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 13$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 13 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 12$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 12 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 11$).
1, 2, 3, 4, 5, 6, 7, 8, 10, 11 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 9$, $i_3 = 10$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 19 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 18$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 18 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 17$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 17 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 16$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 16 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 15$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 15 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 14$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 14 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 13$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 13 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 12$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 12 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 11$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 11 ($i_1 = 2$, $i_2 = 5$, $i'_3 = 8$, $i_3 = 10$).
1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (non-Weierstrass point).

3. The Weierstrass semigroups at points on a plane curve of degree $\leqq 6$ fixed by an involution

In the following two sections a *plane curve* means a smooth irreducible plane curve over an algebraically closed field k of characteristic 0. Let C be a plane curve of degree $d \geqq 4$ with an involution ι and $\pi : C \longrightarrow C_0 = C/\langle\iota\rangle$ the double covering. It is well known that the number of the fixed point under the involution ι is d (resp. $d+1$) if d is even (resp. odd) (for example, see Kikuchi [3]). Hence, the genus $g(C_0)$ of C_0 is $\frac{(d-2)^2}{4}$ (resp. $\frac{(d-2)^2-1}{4}$) if d is even (resp. odd). We set

$$HD(\text{Plane})_d = \big\{H \bigm| \exists \text{ a double covering } \pi : C \to C_0 \text{ with a ramification point } P \text{ where } C \text{ is a plane curve of degree } d \text{ such that } H(P) = H\big\}.$$

For any numerical semigroup H we denote by $d_2(H)$ the numerical semigroup whose elements are $\frac{h}{2}$ with even $h \in H$. For any nonnegative integer g, $\mathcal{H}(g)$ denotes the set of numerical semigroups of genus g. Then we have

$$d_2\big(HD(\text{Plane})_d\big) \subseteqq \mathcal{H}\left(\frac{(d-2)^2}{4}\right)$$

if d is even and

$$d_2\big(HD(\text{Plane})_d\big) \subseteqq \mathcal{H}\left(\frac{(d-2)^2-1}{4}\right)$$

if d is odd. We want to determine the sets $HD(\text{Plane})_d$ and $d_2(HD(\text{Plane})_d)$ for $d \leqq 6$ in this section. In the case $d=4$ we obtain the following:

Proposition 3.1. *We have*

$$HD(\text{Plane})_4 = \big\{\langle 3,4\rangle, \langle 4,5,6,7\rangle\big\} \quad \textit{and} \quad d_2\big(HD(\text{Plane})_4\big) = \big\{\langle 2,3\rangle\big\}.$$

Proof. Let C be a plane curve defined by $yz^3 - x^4 + y^4 = 0$. The involution ι sends (x,y,z) to $(-x,y,z)$. Let us take the point $P = (0,0,1) \in C$, which is fixed by the involution ι. By Section 2 we get $H(P) = \langle 3,4\rangle$.

Let C be a plane curve defined by $yz^3 - x^2z^2 + x^4 + y^4 = 0$. The involution ι sends (x,y,z) to $(-x,y,z)$. Then $y=0$ is the tangent line at $P=(0,0,1) \in C$ with multiplicity 2. Hence we get $H(P) = \langle 4,5,6,7\rangle$. Since $4 \notin \langle 3,5,7\rangle$, we have $\langle 3,5,7\rangle \notin HD(\text{Plane})_4$. Hence we obtain our desired result. □

We also get the result for the case $d=5$ as follows:

Theorem 3.2. *We have*

$$HD(\text{Plane})_5 = \big\{\langle 4,5\rangle, \langle 4,7,10,13\rangle, \langle 6,8,9,10,11,13\rangle, \langle 6,7,8,10,11\rangle, \langle 6,7,8,9,10\rangle\big\}$$

and

$$d_2\big(HD(\text{Plane})_5\big) = \big\{\langle 2,5\rangle, \langle 3,4,5\rangle\big\}.$$

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Proof. Let C be a plane curve defined by $yz^4 - x^5 + y^5 = 0$. The involution ι sends (x, y, z) to $(-x, -y, z)$. Let us take the point $P = (0, 0, 1) \in C$, which is fixed by the involution ι. By Section 2 we get $H(P) = \langle 4, 5\rangle$.

Let C be a plane curve defined by $yz^4 - x^4z + y^5 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. By Section 2 we get $H(P) = \langle 4, 7, 10, 13\rangle$ where $P = (0, 0, 1)$.

Let C be a plane curve defined by $yz^4 - x^3z^2 + x^5 + y^5 = 0$. The involution ι sends (x, y, z) to $(-x, -y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 3. Hence we get $H(P) = \langle 6, 8, 9, 10, 11, 13\rangle$.

Let C be a plane curve defined by $(yz - x^2)(z^3 + x^2z + y^3) + y^4z = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Let us take the point $P = (0, 0, 1) \in C$. Then we get $H(P) = \langle 6, 7, 8, 10, 11\rangle$.

Let C be a plane curve defined by $(yz - x^2)(z^3 + x^2z + y^3) + y^5 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Let us take the point $P = (0, 0, 1) \in C$. Then we get $H(P) = \langle 6, 7, 8, 9, 10\rangle$.

Since $10 \notin \langle 6, 7, 8, 9, 11\rangle$, $8 \notin \langle 6, 7, 9, 10, 11\rangle$ and $4, 6 \notin \langle 7, 8, 9, 10, 11, 12, 13\rangle$, we have proved the statement. □

For the case $d = 6$ we have the following result:

Theorem 3.3. *We obtain*

$$d_2\big(HD(\text{Plane})_6\big) = \big\{\langle 3, 5\rangle, \langle 4, 5, 6\rangle, \langle 4, 5, 7\rangle, \langle 4, 6, 7, 9\rangle, \langle 5, 6, 7, 8, 9\rangle\big\}$$

and

$$\begin{aligned} HD(\text{Plane})_6 = \big\{&\langle 5, 6\rangle, \langle 8, 9, 10, 11, 12\rangle, \langle 8, 9, 10, 14, 15, 21\rangle, \\ &\langle 8, 11, 12, 14, 15, 17, 18, 21\rangle, \langle 8, 12 \to 15, 17, 18, 19\rangle, \langle 10 \to 18\rangle, \\ &\langle 10 \to 16, 18, 19\rangle, \langle 10 \to 14, 16 \to 19\rangle, \langle 10, 11, 12, 14 \to 19\rangle, \langle 10, 12 \to 19, 21\rangle\big\}. \end{aligned}$$

Proof. First, we note that

$$\mathcal{H}(4) = \big\{\langle 2, 9\rangle, \langle 3, 5\rangle, \langle 3, 7, 8\rangle, \langle 4, 5, 6\rangle, \langle 4, 5, 7\rangle, \langle 4, 6, 7, 9\rangle, \langle 5, 6, 7, 8, 9\rangle\big\}.$$

By the list in Section 2 there is no point on a plane curve of degree 6 whose Weierstrass semigroup contains 4. Hence we get $\langle 2, 9\rangle \notin d_2(HD(\text{Plane})_6)$.

Let C be a plane curve defined by $yz^5 - x^6 + y^6 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 6. By Section 2 we get $H(P) = \langle 5, 6\rangle$. Then $d_2(\langle 5, 6\rangle) = \langle 3, 5\rangle$.

By the list in Section 2 there is no point on a plane curve of degree 6 whose Weierstrass semigroup contains 6 and 14. Hence we get $\langle 3, 7, 8\rangle \notin d_2(HD(\text{Plane})_6)$.

Let C be a plane curve defined by $yz^5 - x^4z^2 + x^6 + y^6 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 4. Hence we get $H(P) = \langle 8, 11, 12, 14, 15, 17, 18, 21\rangle$. Then $d_2(\langle 8, 11, 12, 14, 15, 17, 18, 21\rangle) = \langle 4, 6, 7, 9\rangle$.

Let C be a plane curve defined by $(yz - x^2)(z^4 + x^4 + y^4) + y^6 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C at P with multiplicity 12. Hence we get $H(P) = \langle 8, 9, 10, 11, 12\rangle$. Then $d_2(\langle 8, 9, 10, 11, 12\rangle) = \langle 4, 5, 6\rangle$.

Let C be a plane curve defined by $(yz - x^2)(z^4 + x^4 + y^4) + y^6 = 0$. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C at P with multiplicity 10. Hence we get $H(P) = \langle 8, 9, 10, 14, 15, 21\rangle$. Then $d_2(\langle 8, 9, 10, 14, 15, 21\rangle) = \langle 4, 5, 7\rangle$.

Let C be a plane curve defined by

$$a\big(yz^2 - x^2\big)z^4 + b\big(yz - x^2\big)\big(x^4 + y^4\big) + cy^4z^2 + dy^6 = 0,$$

where a, b, c and d are general. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C at P with multiplicity 8. Then we get

$$H(P) = \langle 8, 12 \to 15, 17, 18, 19\rangle \quad \text{and} \quad d_2\big(\langle 8, 12 \to 15, 17, 18, 19\rangle\big) = \langle 4, 6, 7, 9\rangle.$$

Let C be a plane curve defined by

$$a\big(yz^2 - x^2z - y^3\big)z^3 + b\big(yz^2 - x^2z - y^3\big)y^3 + c\big(yz - x^2\big)^3 + d\big(yz - x^2\big)y^2z^2 = 0,$$

where a, b, c and d are general. The involution ι sends (x, y, z) to $(-x, y, z)$. Then $y = 0$ is the tangent line at $P = (0, 0, 1) \in C$ with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C at P with multiplicity 6. The cubic $yz^2 - x^2z - y^3 = 0$ intersects the curve C at P with multiplicity 10. Hence we get $H(P) = \langle 10, 12 \to 19, 21\rangle$. Then $d_2(\langle 10, 12 \to 19, 21\rangle) = \langle 5, 6, 7, 8, 9\rangle$.

Let C be a plane curve defined by

$$a\big(yz^2 - x^2z - y^3\big)z^3 + b\big(yz^2 - x^2z - y^3\big)y^3 + c\big(yz - x^2\big)^3 + d\big(yz - x^2\big)^2z^2 = 0,$$

where a, b, c and d are general. Let us take the point $P = (0, 0, 1) \in C$. Similarly we get

$$H(P) = \langle 10, 11, 12, 14 \to 19\rangle \quad \text{and} \quad d_2\big(\langle 10, 11, 12, 14 \to 19\rangle\big) = \langle 5, 6, 7, 8, 9\rangle.$$

Let C be a plane curve defined by

$$a\big(yz^2 - x^2z - y^3\big)z^3 + b\big(yz^2 - x^2z - y^3\big)y^3 + c\big(yz - x^2\big)^3 + d\big(yz - x^2\big)^2yz = 0,$$

where a, b, c and d are general. Let us take the point $P = (0, 0, 1) \in C$. Similarly we get

$$H(P) = \langle 10 \to 14, 16 \to 19\rangle \quad \text{and} \quad d_2\big(\langle 10 \to 14, 16 \to 19\rangle\big) = \langle 5, 6, 7, 8, 9\rangle.$$

Let C be a plane curve defined by

$$a\big(yz^2 - x^2z - y^3\big)z^3 + b\big(yz^2 - x^2z - y^3\big)y^3 + c\big(yz - x^2\big)^3 + d\big(yz - x^2\big)^2y^2 = 0,$$

where a, b, c and d are general. Let us take the point $P = (0, 0, 1) \in C$. Similarly we get

$$H(P) = \langle 10 \to 16, 18, 19\rangle \quad \text{and} \quad d_2\big(\langle 10 \to 16, 18, 19\rangle\big) = \langle 5, 6, 7, 8, 9\rangle.$$

Let C be a plane curve defined by

$$a\big(yz^2 - x^2z - y^3\big)z^3 + b\big(yz^2 - x^2z - y^3\big)y^3 + c\big(yz - x^2\big)^3 = 0,$$

where a, b, c and d are general. Let us take the point $P = (0, 0, 1) \in C$. Similarly we get

$$H(P) = \langle 10 \to 18\rangle \quad \text{and} \quad d_2\big(\langle 10 \to 18\rangle\big) = \langle 5, 6, 7, 8, 9\rangle. \qquad \square$$

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4. The Weierstrass semigroups on the quotient curve of a plane curve of degree 7 by an involution

In this section we prove the following:

Theorem 4.1. *We have*

$$d_2\big(HD(\text{Plane})_7\big) = \big\{\langle 3,7\rangle\langle 3,8,13\rangle, \langle 5,6,7\rangle, \langle 5,6,9,13\rangle, \langle 5,7,9,11,13\rangle,$$
$$\langle 5,8,9,11,12\rangle, \langle 6,7,8,9,10\rangle, \langle 6,7,8,9,11\rangle, \langle 6,7,8,10,11\rangle,$$
$$\langle 6,7,9,10,11\rangle, \langle 6,8,9,10,11,13\rangle, \langle 7,8,9,10,11,12,13\rangle\big\}.$$

Proof. A plane curve C of degree 7 is 6-gonal, so there is no Weierstrass 4-semigroup on C where an n-semigroup means a numerical semigroup whose least nonzero element is n. Hence the 2-semigroup $\langle 2,13\rangle$ is not contained in $d_2(HD(\text{Plane})_7)$. Moreover, by Coppens [1] there are no pencils g^1_8 on a plane curve of degree 7. Thus, the set $d_2(HD(\text{Plane})_7)$ does not contain any 4-semigroup.

Let C be a plane curve defined by $yz^6 - x^7 + y^7 = 0$. The involution ι sends (x,y,z) to $(-x,-y,z)$. Let us take the point $P = (0,0,1) \in C$, which is fixed by the involution ι. Then $y = 0$ is the tangent line at P with multiplicity 7. Hence we get $H(P) = \langle 6,7\rangle$. Then $d_2(\langle 6,7\rangle) = \langle 3,7\rangle$.

Let C be a plane curve defined by $yz^6 - x^6z + y^7 = 0$. The involution ι sends (x,y,z) to $(-x,y,z)$. Let us take the point $P = (0,0,1) \in C$. Then $y = 0$ is the tangent line at P with multiplicity 6. Hence we get $H(P) = \langle 6,11,16,21,26,31\rangle$. Then $d_2(\langle 6,11,16,21,26,31\rangle) = \langle 3,8,13\rangle$.

We will prove that $\langle 3,10,11\rangle \notin d_2(HD(\text{Plane})_7)$. On the contrary we assume that (C,P) is a plane pointed curve of degree 7 such that $d_2(H(P)) = \langle 3,10,11\rangle$. Let i_1 be the intersection multiplicity of C and the tangent line at P. If $i_1 \leqq 5$, then 6 is a gap at P, which implies that $d_2(H(P)) \not\ni 3$. This is a contradiction. Hence we have $i_1 = 6, 7$. If $i_1 = 7$ (resp. 6), then $H(P) = \langle 6,7\rangle$ (resp. $\langle 6,11,16,21,26,31\rangle$). This is impossible.

Let C be a plane curve defined by $a(yz^4 - x^5)z^2 + b(yz^6 - x^7) + cy^7 = 0$ where a, b and c are general. The involution ι sends (x,y,z) to $(-x,-y,z)$. Then $y = 0$ is the tangent line at $P = (0,0,1)$ with multiplicity 5. Hence we get $H(P) = \langle 10,14,15,18,19,22,23,26,27,31\rangle$. Then we have $d_2(H(P)) = \langle 5,7,9,11,13\rangle$.

Let C be a plane curve defined by $a(yz^3 - x^4)z^3 + b(yz^3 - x^4)(x^2z + y^3) + cy^7 = 0$ where a, b and c are general. The involution ι sends (x,y,z) to $(-x,y,z)$. Then $y = 0$ is the tangent line at $P = (0,0,1)$ with multiplicity 4. Moreover, the quartic curve $yz^3 - x^4 = 0$ intersects the curve C at P with multiplicity 28. The gap sequence at P is $1 \to 11, 13, 14, 17, 29$. Thus, $d_2(H(P)) = \langle 6,8,9,10,11,13\rangle$.

We will show that

$$\langle 5,6,7\rangle, \langle 5,6,9,13\rangle, \langle 5,8,9,11,12\rangle \in d_2\big(HD(\text{Plane})_7\big).$$

Let C_j be a plane curve defined by

$$a\big(yz^6 - x^2z^5\big) + b\big(yz - x^2\big)\big(z^5 + x^4z + y^5\big) + cy^jz^{7-j} = 0$$

for $j = 5,6,7$ where a, b and c are general. The involution ι sends (x,y,z) to $(-x,y,z)$. Let us take the point $P_j = (0,0,1) \in C_j$. Then $y = 0$ is the tangent line at P_j with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C_j at P_j with multiplicity $2j$. Hence, the gap sequence at P_j is $1 \to 9, 2j+1 \to 2j+5, 4j+1$. Thus, we obtain $d_2(H(P_7)) = \langle 5,6,7\rangle$, $d_2(H(P_6)) = \langle 5,6,9,13\rangle$ and $d_2(H(P_5)) = \langle 5,8,9,11,12\rangle$.

Next, we will prove that

$$\langle 5,6,8\rangle, \langle 5,7,8,9\rangle, \langle 5,7,8,11\rangle \notin d_2\big(HD(\text{Plane})_7\big).$$

Let (C,P) be a pointed plane curve of degree 7. If $i_1 = 7,6,4,3$, then 10 is a gap at P. We may exclude these cases. If $i_1 = 5$, then 16 is a gap at P. We do not need to consider this case. Hence, we may assume that $i_1 = 2$. We have $5 \leqq i_2 \leqq 14$. Then the set $G(P)$ of gaps at P contains the set $\{1 \to 9, i_2 + 1 \to i_2 + 5\}$. If $5 \leqq i_2 \leqq 9$, then $G(P) \ni 10$. If $11 \leqq i_2 \leqq 14$, then $G(P) \ni 16$. If $i_2 = 10$, the set $G(P)$ contains both 12 and 14. Therefore, the above three 5-semigroups are not in $d_2(HD(\text{Plane})_7)$.

We will show that

$$\langle 6, 7, 8, 9, 10\rangle, \langle 6, 7, 8, 9, 11\rangle, \langle 6, 7, 8, 10, 11\rangle, \langle 6, 7, 9, 10, 11\rangle$$

belong to $d_2(HD(\text{Plane})_7)$. Let C_{ij} be a plane curve defined by

$$a\big(yz^6 - x^2z^5 + y^3z^4\big) + b\big(yz^2 - x^2z + y^3\big)\big(z^4 + x^4 + y^4\big) + cy^iz^{7-i-2j}\big(yz - x^2\big)^j = 0$$

for $(i, j) = (1, 3), (0, 3), (2, 2), (1, 2)$ where a, b and c are general. The involution ι sends (x, y, z) to $(-x, y, z)$. Let us take the point $P_{ij} = (0, 0, 1) \in C_{ij}$, which is fixed by the involution ι. Then $y = 0$ is the tangent line at P_{ij} with multiplicity 2. Moreover, the conic $yz - x^2 = 0$ intersects the curve C_{ij} with multiplicity 6. The cubic $yz^2 - x^2z + y^3$ intersects the curve C_{ij} with multiplicity $2i + 6j$. Hence, the gap sequence at P_{ij} is $1 \to 11, 13, 2i + 6j + 1, 2i + 6j + 2, 2i + 6j + 3$. Thus, we obtain $d_2(H(P_{13})) = \langle 6, 7, 8, 9, 10\rangle$, $d_2(H(P_{03})) = \langle 6, 7, 8, 9, 11\rangle$, $d_2(H(P_{22})) = \langle 6, 7, 8, 10, 11\rangle$, $d_2(H(P_{12})) = \langle 6, 7, 9, 10, 11\rangle$.

Finally, we will show that the semigroup $\langle 7, 8, 9, 10, 11, 12, 13\rangle$ is contained in the set $d_2(HD(\text{Plane})_7)$. Let C be a plane curve defined by

$$\begin{aligned} &a\big(yz - x^2\big)z^5 + b\big(yz - x^2\big)\big(x^2 + y^2\big)z^3 + cy^4z^3 \\ &\quad + d\big(a\big(yz - x^2\big)z^2 + b\big(yz - x^2\big)\big(x^2 + y^2\big) + cy^4\big)\big(x^2z + y^3\big) + ey^7 = 0, \end{aligned}$$

where a, b, c, d and e are general. The involution ι sends (x, y, z) to $(-x, y, z)$. Let us take the point $P = (0, 0, 1) \in C$. Then $y = 0$ is the tangent line at P with multiplicity 2. The conic $yz - x^2 = 0$ intersects the curve C with multiplicity 8. Moreover, the quartic curve $a(yz - x^2)z^2 + b(yz - x^2)(x^2 + y^2) + cy^4 = 0$ intersects the curve C with multiplicity 14. Hence, the gap sequence at P is $1 \to 13$, 15, 17. Thus, we have $d_2(H(P)) = \langle 7, 8, 9, 10, 11, 12, 13\rangle$. □

Uncited references

Q1 [4]

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