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RESEARCH ARTICLE

On Weierstrass semigroups of double coverings of genus three curves

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Abstract Let C be a complete non-singular curve of genus 3 over an algebraically closed field of characteristic 0. We determine all possible Wierstrass semigroups of ramification points on double coverings of C whose covering curves have genus greater than 8. Moreover, we construct double coverings with ramification points whose Weierstrass semigroups are the possible ones.

Keywords Weierstrass semigroups of points · Double coverings of curves · Plane curves of degree 4

1 Introduction

Let C be a complete nonsingular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $k(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \text{there exists } f \in k(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of P* where $\mathbb{N}_0$ denotes the additive monoid of non-negative integers.

Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a curve. We are interested in the Weierstrass semigroup of a ramification point $\tilde{P}$ on the double covering π. Such a Weierstrass semigroup $\tilde{H}$ is called the *double covering type*. Let $\tilde{g}$ be the genus of $\tilde{C}$.

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If C is the projective line, then $\tilde{P}$ is a Weierstrass point on the hyperelliptic curve $\tilde{C}$. Hence, the semigroup $H(\tilde{P})$ is generated by 2 and $2\tilde{g}+1$. If C is an elliptic curve, then the semigroup $H(\tilde{P})$ is either $\langle 3,4,5\rangle$ or $\langle 3,4\rangle$ or $\langle 4,5,6,7\rangle$ or $\langle 4,6,2\tilde{g}-3\rangle$ with $\tilde{g} \geqq 4$ or $\langle 4,6,2\tilde{g}-1,2\tilde{g}+1\rangle$ with $\tilde{g} \geqq 4$ where for any positive integers $a_1, a_2, \ldots, a_n$ we denote by $\langle a_1, a_2, \ldots, a_n\rangle$ the semigroup generated by $a_1, a_2, \ldots, a_n$. Conversely, there is a double covering of an elliptic curve with a ramification point whose Weierstrass semigroup is any semigroup in the above ones (for example, see [1, 2]). On the other hand, Oliveira and Pimentel [6] studied about Weierstrass semigroups $H(\tilde{P})$ in the case where the genus of C is 2. They showed that for a semigroup $\tilde{H} = \langle 6,8,10,n\rangle$ with an odd number $n \geqq 11$ there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$. Moreover, in [2] we determine the Weierstrass semigroups of ramification points on double coverings of curves of genus 2 where the genus of the covering curve is larger than or equal to 6.

A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is a finite set. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, which is denoted by $g(H)$. It is known that the Weierstrass semigroup of a point on a curve of genus g is a numerical semigroup of genus g. For a numerical semigroup $\tilde{H}$ we denote by $d_2(\tilde{H})$ the set of consisting of the elements $\tilde{h}/2$ with even $\tilde{h} \in \tilde{H}$, which becomes a numerical semigroup. In this paper we will study the Weierstrass semigroups of ramification points on double coverings of curves of genus 3, namely we prove

Main Theorem *Let $\tilde{H}$ be a numerical semigroup of genus $\geqq 9$ with $g(d_2(\tilde{H})) = 3$. Then $\tilde{H}$ is the double covering type.*

We note that Oliveira, Torres and Villanueva [7] determine the Weierstrass semigroups of ramification points on some kinds of double coverings of curves of genus 3 (see Remarks 3.2, 5.4 and 6.2).

2 Weierstrass semigroups on double coverings

First, we give a relation between the Weierstrass semigroups of $\tilde{P}$ and $\pi(\tilde{P})$ where $\tilde{P}$ is a ramification point of a double covering π.

Remark 2.1 Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of a curve with a ramification point $\tilde{P}$. Then we have $H(\pi(\tilde{P})) = d_2(H(\tilde{P}))$. (For example, see [8].)

Second, we state the main tool for constructing double coverings with ramification points whose Weierstrass semigroups are the desired ones.

Theorem 2.2 *Let H and $\tilde{H}$ be numerical semigroups with $d_2(\tilde{H}) = H$. We set $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h}$ is odd$\}$ and $g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - r$. Assume that H is Weierstrass. Take a pointed curve (C, P) with $H(P) = H$. Let $Q_1, \ldots, Q_r$ be points of C different from P with $h^0(Q_1 + \cdots + Q_r) = 1$. Moreover, assume that $\tilde{H}$ has an expression*

$$\tilde{H} = 2H + \langle n, n+2l_1, \ldots, n+2l_s\rangle$$

of generators with positive integers $l_1, \ldots, l_s$ *such that*

$$h^0(l_i P + Q_1 + \cdots + Q_r) = h^0((l_i - 1)P + Q_1 + \cdots + Q_r) + 1$$

for all i. *If the complete linear system* $|nP - 2Q_1 - \cdots - 2Q_r|$ *is base point free, then there is a double covering* $\pi : \tilde{C} \longrightarrow C$ *with a ramification point* $\tilde{P}$ *satisfying* $H(\tilde{P}) = \tilde{H}$, *hence* $\tilde{H}$ *is the double covering type.*

Proof We consider the divisor $D = \frac{n+1}{2}P - Q_1 - \cdots - Q_r$ on C. Then the complete linear system $|2D - P| = |nP - 2Q_1 - \cdots - 2Q_r|$ is base point free by the assumption. Hence, we have $2D - P \sim R_1 + \cdots + R_{n-2r}$ where R_i's are distinct points different from P. We set $\mathcal{L} = \mathcal{O}_C(-D)$. Then there is an isomorphism $\mathcal{L}^{\otimes 2} \overset{\phi}{\cong} \mathcal{O}_C(-(P + R_1 + \cdots + R_{n-2r})) \subset \mathcal{O}_C$. Hence, the direct sum $\mathcal{O}_C \oplus \mathcal{L}$ has an $\mathcal{O}_C$-algebra structure through ϕ. Let $\pi : \tilde{C} = \text{Spec}\ (\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ be a canonical morphism. We note that the genus $g(\tilde{C})$ of $\tilde{C}$ is $2g(H) + \frac{n-1}{2} - r$, because the branch divisor of π is $P + R_1 + \cdots + R_{n-2r}$. Let $\tilde{P} \in \tilde{C}$ such that $\pi(\tilde{P}) = P$. By Proposition 2.1 in [3] we have

$$\begin{aligned} h^0((n-1)\tilde{P}) &= h^0\left(\frac{n-1}{2}P\right) + h^0\left(\frac{n-1}{2}P - D\right) \\ &= h^0\left(\frac{n-1}{2}P\right) + h^0(Q_1 + \cdots + Q_r - P) = h^0\left(\frac{n-1}{2}P\right), \end{aligned}$$

because we have $h^0(Q_1 + \cdots + Q_r) = 1$ and $P \neq Q_i$ for all i. Moreover, we have

$$h^0((n+1)\tilde{P}) = h^0\left(\frac{n+1}{2}P\right) + h^0(Q_1 + \cdots + Q_r) = h^0\left(\frac{n+1}{2}P\right) + 1.$$

If $n + 1 \in H(\tilde{P})$, then $\frac{n+1}{2} \in H(P)$, hence we have $h^0(\frac{n+1}{2}P) = h^0(\frac{n-1}{2}P) + 1$. Thus, we get $h^0((n+1)\tilde{P}) = h^0((n-1)\tilde{P}) + 2$, which implies that $n \in H(\tilde{P})$. If $n+1 \notin H(\tilde{P})$, then $\frac{n+1}{2} \notin H(P)$, hence we have $h^0(\frac{n+1}{2}P) = h^0(\frac{n-1}{2}P)$. Thus, we get $h^0((n+1)\tilde{P}) = h^0((n-1)\tilde{P}) + 1$, which implies that $n \in H(\tilde{P})$. Moreover, for any $i = 1, \ldots, s$ we have

$$h^0((n+2l_i-1)\tilde{P}) = h^0\left(\frac{n+2l_i-1}{2}P\right) + h^0((l_i-1)P + Q_1 + \cdots + Q_r)$$

and

$$h^0((n+2l_i+1)\tilde{P}) = h^0\left(\frac{n+2l_i+1}{2}P\right) + h^0(l_i P + Q_1 + \cdots + Q_r).$$

By the assumption we obtain $n + 2l_i \in H(\tilde{P})$. Hence, by the above we get

$$H(\tilde{P}) \supseteqq 2H(P) + \langle n, n+2l_1, \ldots, n+2l_s \rangle = \tilde{H}.$$

Since $g(H(\tilde{P})) = g(\tilde{C}) = g(\tilde{H})$, we get $H(\tilde{P}) = \tilde{H}$. □

For a numerical semigroup H we set $c(H) = \min\{c \in \mathbb{N}_0 | c + \mathbb{N}_0 \subseteqq H\}$. A numerical semigroup H is called an *m-semigroup* when m is the minimum positive integer in H. For an m-semigroup H the set $S(H) = \{m, s_1, s_2, \dots, s_{m-1}\}$ with $s_i = \min\{h \in H \mid h \equiv i \mod m\}$ for all i is called the *standard basis* for H.

Lemma 2.3 *Let H be an m-semigroup and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. We set $n = \min\{\tilde{h} \in \tilde{H} | \tilde{h}$ is odd$\}$. If $n \geqq 2c(H) - 1$ and $n \neq 2m - 1$, then we get*

$$g(H) + \frac{n-1}{2} \leqq g(\tilde{H}) \leqq 2g(H) + \frac{n-1}{2}.$$

The right-hand (resp. left-hand) equality of the above inequalities holds if and only if $\tilde{H} = 2H + n\mathbb{N}_0$ (resp. $2H + n\mathbb{N}_0 + (n+2)\mathbb{N}_0 + (n+4)\mathbb{N}_0 + \cdots + (n+2(m-1))\mathbb{N}_0$).

Proof Since $\tilde{H} \supseteqq 2H + n\mathbb{N}_0$, by Lemma 2.1 in [4] we obtain

$$g(\tilde{H}) \leqq g(2H + n\mathbb{N}_0) = 2g(H) + \frac{n-1}{2}.$$

Moreover, we have

$$S(2H + n\mathbb{N}_0) = \{2m, 2s_1, \dots, 2s_{m-1}, n, n + 2s_1, \dots, n + 2s_{m-1}\}.$$

By the definition of n we obtain

$$\tilde{H} \subseteqq 2H + n\mathbb{N}_0 + (n+2)\mathbb{N}_0 + (n+4)\mathbb{N}_0 + \cdots + (n+2(m-1))\mathbb{N}_0,$$

which implies that

$$g(\tilde{H}) \geqq g(H) + \sum_{i=0}^{m-1} \left[\frac{n+2i}{2m}\right].$$

We set $n = 2mq + r$ with $q \in \mathbb{N}_0$ and $1 \leqq r \leqq 2m - 1$. Then we have

$$\sum_{i=0}^{m-1} \left[\frac{n+2i}{2m}\right] = mq + \sum_{i=0}^{m-1} \left[\frac{r+2i}{2m}\right] = mq + \frac{r + 2(m-1) - (2m-1)}{2}$$
$$= mq + \frac{r-1}{2} = mq + \frac{n - 2mq - 1}{2} = \frac{n-1}{2}. \qquad \square$$

When $\tilde{H}$ satisfies the equality of the left-hand inequality in Lemma 2.3, we get the following:

Proposition 2.4 *Let H be a Weierstrass numerical semigroup and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. We set $n = \min\{h \in \tilde{H} | h$ is odd$\}$. Assume that $g(\tilde{H}) = g(H) + \frac{n-1}{2}$ and $n \geqq 4g(H) + 1$. Then $\tilde{H}$ is the double covering type.*

Proof Let (C, P) be a pointed curve with $H(P) = H$. Let $Q_1, \dots, Q_g$ be points of C distinct from P with $h^0(Q_1 + \cdots + Q_g) = 1$. We set $D = \frac{n+1}{2}P - Q_1 - \cdots - Q_g$.

Then by the assumption we have $\deg(2D-P)=n-2g \geqq 4g+1-2g=2g+1$, which implies that the complete linear system $|2D-P|$ is very ample. By the proof of Theorem 2.2 we get a double covering $\pi:\tilde{C}\longrightarrow C$ with a ramification point $\tilde{P}$ over P. In this case we obtain $g(H(\tilde{P}))=g(\tilde{C})=g(H)+\frac{n-1}{2}$. By the assumption $g(\tilde{H})=g(H)+\frac{n-1}{2}$ and Lemma 2.3 we get $\tilde{H}=H(\tilde{P})$. □

When $\tilde{H}$ satisfies the equality of the right-hand inequality in Lemma 2.3, by [4] we have the following:

Remark 2.5 Let $\tilde{H}$ be a numerical semigroup such that $H=d_2(\tilde{H})$ is Weierstrass. We set $n=\min\{h\in\tilde{H}\,|\,h \text{ is odd}\}$. Assume that $n\geqq 2c(H)-1$ and $n\neq 2m-1$ where H is an m-semigroup. If $g(\tilde{H})=2g(H)+\frac{n-1}{2}$, then $\tilde{H}$ is the double covering type.

3 Case $g(\tilde{H})=6+\frac{n-1}{2}$

A numerical semigroup H of genus 3 is either $\langle 3,4\rangle$ or $\langle 3,5,7\rangle$ or $\langle 4,5,6,7\rangle$. In this section we use the following notations and assumptions: Let $\tilde{H}$ be a numerical semigroup with $d_2(\tilde{H})=H$ and n the least odd number in $\tilde{H}$. Assume that $n\geqq 2c(H)-1$ and $n\neq 7$. By Lemma 2.3 we obtain

$$3+\frac{n-1}{2}\leqq g(\tilde{H})\leqq 6+\frac{n-1}{2}.$$

In this section we are in the case $g(\tilde{H})=6+\frac{n-1}{2}$. By Lemma 2.3 and Remark 2.5 we get the following:

Proposition 3.1 *$\tilde{H}$ is the double covering type. In this case we have $\tilde{H}=2H+n\mathbb{N}_0$.*

Remark 3.2 Proposition 3.1 is proved in [7] using Theorem 2.2 in [4]. The method is the same as ours.

4 Case $g(\tilde{H})=5+\frac{n-1}{2}$

Let H be a numerical semigroup of genus 3. In this section let $\tilde{H}$ be a numerical semigroup with $d_2(\tilde{H})=H$ and n the least odd number in $\tilde{H}$. Moreover, we assume that $g(\tilde{H})=5+\frac{n-1}{2}$ with $n\geqq\max\{9,2c(H)-1\}$.

Proposition 4.1 *Let $H=\langle 3,4\rangle$. Then we have $\tilde{H}=\langle 6,8,n,n+10\rangle$, which is the double covering type.*

Proof Since $S(2H+n\mathbb{N}_0)=\{6,8,16,n,n+8,n+16\}$, we get $\tilde{H}=\langle 6,8,n,n+10\rangle$ by the assumption on $g(\tilde{H})$. Let (C,P) be a pointed curve with $H(P)=\langle 3,4\rangle$. Then for any point Q of C we have $h^0(5P+Q)=h^0(4P+Q)+1$. By Theorem 2.2 $\tilde{H}$ is the double covering type. □

In the case where $H=\langle 3,5,7\rangle$ we also obtain the following:

Proposition 4.2 *Let* $H=\langle 3,5,7\rangle$. *Then* $\tilde{H}$ *is either* $\langle 6,10,14,n,n+8\rangle$ *or* $\langle 6,10,14,n,n+4\rangle$. *Moreover,* $\tilde{H}$ *is the double covering type.*

Proof Because of $S(2H+n\mathbb{N}_0)=\{6,10,14,n,n+10,n+14\}$ $\tilde{H}$ is either $\langle 6,10,14,n,n+8\rangle$ or $\langle 6,10,14,n,n+4\rangle$. Let (C,P) be a pointed plane curve of degree 4 with $H(P)=\langle 3,5,7\rangle$. We denote by T_P the tangent line on C at P. Then the intesection $C.T_P$ is equal to $3P+Q$ with $Q\neq P$ which is a canonical divisor on C. Hence we get $h^0(2P+Q)=h^0(P+Q)+1$. By Theorem 2.2 $\langle 6,10,14,n,n+4\rangle$ is the double covering type. Let us take a point Q' on C with $Q'\neq Q$. Since $3P+Q'$ is not a canonical divisor, we get $h^0(4P+Q')=h^0(3P+Q')+1$, which implies that $\langle 6,10,14,n,n+8\rangle$ is the double covering type. □

Proposition 4.3 *Let* $H=\langle 4,5,6,7\rangle$. *Then* $\tilde{H}$ *is either* $\langle 8,10,12,14,n,n+6\rangle$ *or* $\langle 8,10,12,14,n,n+4\rangle$ *or* $\langle 8,10,12,14,n,n+2\rangle$, *which is the double covering type.*

Proof Since $S(2H+n\mathbb{N}_0)=\{8,10,12,14,n,n+10,n+12,n+14\}$, $\tilde{H}$ is one of the three semigroups in the statement. Let (C,P) be a pointed plane curve of degree 4 with $H(P)=\langle 4,5,6,7\rangle$. We denote by T_P the tangent line on C at P. Then the intesection $C.T_P$ is equal to $2P+Q_1+Q_2$ with $Q_i\neq P$ which is a canonical divisor on C. Let Q be a point of C distinct from P, Q_1 and Q_2. Then the divisor $3P+Q$ is not canonical, which implies that $h^0(3P+Q)=2$. Moreover, we get

$$h^0(2P+Q)=1+h^0(Q_1+Q_2-Q)=1.$$

Hence, the semigroup $\langle 8,10,12,14,n,n+6\rangle$ is the double covering type. On the other hand, we have $h^0(2P+Q_i)=2$ and $h^0(P+Q_i)=1$. Thus, the semigroup $\langle 8,10,12,14,n,n+4\rangle$ is the double covering type.

Let us take a hyperelliptic curve C of genus 3 with an ordinary point P. Then there is a unique point Q with $h^0(P+Q)=2$. Hence, the semigroup $\langle 8,10,12,14,n,n+2\rangle$ is the double covering type. □

5 Case $g(\tilde{H})=4+\frac{n-1}{2}$

First, we make a remark about $h^0(P+Q_1+Q_2)$ for three points P,Q_1,Q_2 on a plane curve of degree 4.

Lemma 5.1 *Let C be a plane curve of degree* 4. *For three distinct points* P,Q_1,Q_2 *on C we have* $h^0(P+Q_1+Q_2)=2$ *if* P,Q_1,Q_2 *are on a line. Otherwise, we get* $h^0(P+Q_1+Q_2)=1$.

Proof Let P, Q_1 and Q_2 be on a line, that is to say, $P + Q_1 + Q_2 + R$ for some point R is a canonical divisor K. Then we get $h^0(P + Q_1 + Q_2) = 2$. If P, Q_1, Q_2 are not on any line, then $h^0(K - P - Q_1 - Q_2) = 0$, which implies that $h^0(P + Q_1 + Q_2) = 1$. □

Remark 5.2 Let C be a plane curve of degree 4 with three distinct points P, Q_1, Q_2. Let n be an odd number larger than or equal to 11. We set $D = \frac{n+1}{2}P - Q_1 - Q_2$. Then by the method of the proof of Theorem 2.2 we get a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P. In this case n is the minimum of the odd numbers in $H(\tilde{P})$. Moreover, we have

$$h^0((n+3)\tilde{P}) = h^0\left(\frac{n+3}{2}P\right) + h^0(P + Q_1 + Q_2).$$

By Lemma 5.1 if P, Q_1, Q_2 are on a line, then $n + 2 \in H(\tilde{P})$. Otherwise, $n + 2 \notin H(\tilde{P})$.

From here in this section, let $\tilde{H}$ be a numerical semigroup with $d_2(\tilde{H}) = H$ and n the least odd number in $\tilde{H}$. Moreover, we assume that $g(\tilde{H}) = 4 + \frac{n-1}{2}$ with $n \geqq 11$.

Proposition 5.3 *Let* $H = \langle 3, 4\rangle$. *Then* $\tilde{H}$ *is either* $\langle 6, 8, n, n+4\rangle$ *or* $\langle 6, 8, n, n+2\rangle$, *which is the double covering type.*

Proof It is trivial that $\tilde{H}$ is one of the above two semigroups. Let P be a point on a plane curve C of degree 4 with $H(P) = \langle 3, 4\rangle$. We take distinct points Q_1 and Q_2 different from P such that P, Q_1 and Q_2 are on a line. Then we get $H(\tilde{P}) = \langle 6, 8, n, n+2\rangle$ where $\tilde{P}$ is as in Remark 5.2.

We take distinct points Q_1 and Q_2 different from P such that P, Q_1 and Q_2 are not on any line. We obtain $H(\tilde{P}) = \langle 6, 8, n, n+4\rangle$. □

Remark 5.4 It is showed in [7] using the equations defining the function fields corresponding to double coverings that $\langle 6, 8, n, n+4\rangle$ with $n \equiv 4 \bmod 3$ and $n \geqq 11$ is the double covering type.

Proposition 5.5 *Let* $H = \langle 3, 5, 7\rangle$. *Then* $\tilde{H}$ *is either* $\langle 6, 10, 14, n, n+4, n+8\rangle$ *or* $\langle 6, 10, 14, n, n+2\rangle$, *which is the double covering type.*

Proof We see easily that $\tilde{H}$ is one of the above two numerical semigroups. Let P be a point of a plane curve C of degree 4 with $H(P) = \langle 3, 5, 7\rangle$ and Q_1, Q_2 distinct points of C different from P. We use the notations in Remark 5.2. If P, Q_1 and Q_2 are not on any line, we have $n + 2 \notin H(\tilde{P})$, which implies that $H(\tilde{P}) = \langle 6, 10, 14, n, n+4, n+8\rangle$. If P, Q_1 and Q_2 are on a line, we obtain $H(\tilde{P}) = \langle 6, 10, 14, n, n+2\rangle$. □

Proposition 5.6 *Let* $H = \langle 4, 5, 6, 7\rangle$. *Then* $\tilde{H}$ *is either* $\langle 8, 10, 12, 14, n, n+4, n+6\rangle$ *or* $\langle 8, 10, 12, 14, n, n+2, n+6\rangle$ *or* $\langle 8, 10, 12, 14, n, n+2, n+4\rangle$, *which is the double covering type.*

Proof In view of $g(\tilde{H}) = 4 + \frac{n-1}{2}$ it is easy to describe the numerical semigroups $\tilde{H}$. Let P be an ordinary point on a plane curve C of degree 4 and Q_1, Q_2 distinct points different from P. Let T_PC be the tangent line at P on C and $T_PC.C$ the intersection of T_PC and C. Let $D = \frac{n+1}{2}P - Q_1 - Q_2$ and the notations as in Remark 5.2. We set $T_PC.C = 2P + R_1 + R_2$ where R_1 and R_2 are points of C distinct from P.

Let $\tilde{H} = \langle 8, 10, 12, 14, n, n+4, n+6\rangle$. Assume that P, Q_1 and Q_2 are not on any line. Then $n + 2 \notin H(\tilde{P})$. Hence we obtain $H(\tilde{P}) = \tilde{H}$.

Let $\tilde{H} = \langle 8, 10, 12, 14, n, n+2, n+6\rangle$. Let $\{Q_1, Q_2\} \neq \{R_1, R_2\}$ such that P, Q_1 and Q_2 are on a line. Then $n + 2 \in H(\tilde{P})$. Moreover, we have

$$h^0((n+5)\tilde{P}) = h^0\left(\frac{n+5}{2}P\right) + h^0(2P + Q_1 + Q_2) = h^0\left(\frac{n+5}{2}P\right) + 2,$$

which implies that $n + 4 \notin H(\tilde{P})$. Hence, we get $H(\tilde{P}) = \tilde{H}$.

Let $\tilde{H} = \langle 8, 10, 12, 14, n, n+2, n+4\rangle$. Let $\{Q_1, Q_2\} = \{R_1, R_2\}$. If $R_1 = R_2$, replacing P by another ordinary point we may assume that $R_1 \neq R_2$. Since P, R_1 and R_2 are on a line, we get $n + 2 \in H(\tilde{P})$. In view of $h^0(2P + Q_1 + Q_2) = h^0(K) = 3$ we obtain $n + 4 \in H(\tilde{P})$. Thus, $H(\tilde{P}) = \tilde{H}$. □

6 Case $g(\tilde{H}) = 3 + \frac{n-1}{2}$

By Proposition 2.4 we get the following:

Proposition 6.1 *Let H be a numerical semigroup of genus 3 and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. We denote by n the least odd number in $\tilde{H}$. Let H be the semigroup $\langle 4, 5, 6, 7\rangle$ (resp. $\langle 3, 5, 7\rangle$, resp. $\langle 3, 4\rangle$) and $n \geqq 13$. If $g(\tilde{H}) = 3 + \frac{n-1}{2}$, then we have $\tilde{H} = \langle 8, 10, 12, 14, n, n+2, n+4, n+6\rangle$ (resp. $\langle 6, 10, 14, n, n+2, n+4\rangle$, resp. $\langle 6, 8, n, n+2, n+4\rangle$), which is the double covering type.*

Remark 6.2 It is proved in [7] using the equations defining the function fields corresponding to double coverings that $\tilde{H} = \langle 8, 10, 12, 14, n, n+2, n+4, n+6\rangle$ with odd $n \geqq 17$ is the double covering type.

7 The proof of Main Theorem

In the last section we will prove the main theorem in this paper using the above propositions.

Lemma 7.1 *Let H be an m-semigroup of genus 3 with $m \neq 2$ and $\tilde{H}$ a numerical semigroup of genus $\geqq 9$ with $d_2(\tilde{H}) = H$. We denote by n the least odd number in $\tilde{H}$. If $n < 2c(H) - 1$ or $n = 2m - 1$, then $\tilde{H}$ is one of the following 5 numerical semigroups $\langle 7, 8, 10, 12\rangle$, $\langle 6, 7, 10\rangle$, $\langle 6, 7, 8\rangle$, $\langle 6, 8, 9\rangle$, $\langle 6, 8, 9, 19\rangle$. Moreover, there is a double covering of a curve of genus 3 with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$.*

Proof In view of $m \neq 2$ the semigroup H is either $\langle 4,5,6,7\rangle$ or $\langle 3,5,7\rangle$ or $\langle 3,4\rangle$.

Let $H = \langle 4,5,6,7\rangle$. If $n = 3$, then $\langle 3,8,10\rangle \subsetneqq \tilde{H}$. Then $g(\tilde{H}) \leqq 5$. If $n = 5$, then $\langle 5,8,12,14\rangle \subsetneqq \tilde{H}$. We get $g(\tilde{H}) \leqq 8$. If $n = 7$, then $\langle 7,8,10,12\rangle \subseteqq \tilde{H}$. Hence, we get $\tilde{H} = \langle 7,8,10,12\rangle$ whose genus is 9.

Let $H = \langle 3,5,7\rangle$. Assume that $n \leqq 5$. Then $\langle 3,10,14\rangle \subsetneqq \tilde{H}$ or $\langle 5,6,14\rangle \subsetneqq \tilde{H}$, which implies that $g(\tilde{H}) \leqq 8$. This is a contradiction. If $n = 7$, then $\langle 6,7,10\rangle \subseteqq \tilde{H}$. In view of $g(\tilde{H}) \geqq 9$ we get $\tilde{H} = \langle 6,7,10\rangle$ whose genus is 9

Let $H = \langle 3,4\rangle$. Assume that $n \leqq 5$. Then $\langle 3,8\rangle \subsetneqq \tilde{H}$ or $\langle 5,6,8\rangle \subsetneqq \tilde{H}$, which implies that $g(\tilde{H}) \leqq 7$. This is a contradiction. If $n = 7$, then $\langle 6,7,8\rangle \subseteqq \tilde{H}$. In view of $g(\tilde{H}) \geqq 9$ we get $\tilde{H} = \langle 6,7,8\rangle$ whose genus is 9. If $n = 9$, then $\langle 6,8,9\rangle \subseteqq \tilde{H}$. In view of $g(\tilde{H}) \geqq 9$ we get $\tilde{H} = \langle 6,8,9\rangle$ whose genus is 10 or $\langle 6,8,9,19\rangle$ whose genus is 9.

Let (C, P) be a pointed plane curve of degree 4 with $H(P) = H$. Let $\tilde{H}$ be either $\langle 7,8,10,12\rangle$ or $\langle 6,7,10\rangle$ or $\langle 6,7,8\rangle$. We set $D = 4P$. In view of $\deg(2D - P) = 7$ by the proof of Theorem 2.2 we can construct a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) \ni 7$. Since the genus of $\tilde{C}$ is 9, the semigroup coincides with $\tilde{H}$. Let $\tilde{H} = \langle 6,8,9\rangle$. We set $D = 5P$. In view of $\deg(2D - P) = 9$ there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) \ni 9$. Since the genus of $\tilde{C}$ is 10, the semigroup $H(\tilde{P})$ coincides with $\tilde{H}$. Let $\tilde{H} = \langle 6,8,9,19\rangle$. We set $D = 5P - Q$ with $Q \neq P$. In view of $\deg(2D - P) = 7$ there is a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) \ni 9$. Since the genus of $\tilde{C}$ is 9, the semigroup $H(\tilde{P})$ should coincide with $\tilde{H}$. □

The proof of Main Theorem We set $H = d_2(\tilde{H})$. Let $H = \langle 2,7\rangle$. In view of $g(\tilde{H}) \geqq 9 > 3$, $\tilde{H}$ is a 4-semigroup. By [5] we gain the desired result.

We assume that $H \neq \langle 2,7\rangle$. We denote by n the least odd number in $\tilde{H}$. Let H be an m-semigroup. By Lemma 7.1 we may assume that $n \geqq 2c(H) - 1$ and $n \neq 2m - 1$. By Lemma 2.3 we have $3 + \frac{n-1}{2} \leqq g(\tilde{H}) \leqq 6 + \frac{n-1}{2}$. By Propositions 3.1, 4.1, 4.2, 4.3, 5.3, 5.5, 5.6 and 6.1 Main Theorem has been proved. □

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