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Existence of the non-primitive Weierstrass gap sequences on curves of genus 8

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Abstract. We show that for any possible Weierstrass gap sequence L on a non-singular curve of genus 8 with twice the smallest positive non-gap is less than the largest gap there exists a pointed non-singular curve (C, P) over an algebraically closed field of characteristic 0 such that the Weierstrass gap sequence at P is L. Combining this with the result in [6] we see that every possible Weierstrass gap sequence of genus 8 is attained by some pointed non-singular curve.

Keywords: Weierstrass semigroup of a point, Double covering of a curve, Cyclic covering of an elliptic curve.

Mathematical subject classification: Primary: 14H55; Secondary: 14H30, 14C20.

1 Introduction

Let C be a complete nonsingular irreducible curve of genus g over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $k(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) = \{\alpha \in \mathbb{N}_0 |\text{ there exists } f \in k(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point* P where $\mathbb{N}_0$ denotes the additive semigroup of non-negative integers. The increasing elements of the complement $\mathbb{N}_0 \backslash H(P)$ of $H(P)$ in $\mathbb{N}_0$ are called the *Weierstrass gap sequence* at P. Then $H(P)$ is a subsemigroup of $\mathbb{N}_0$ with $\sharp(\mathbb{N}_0 \backslash H(P)) = g$.

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Conversely, let H be a subsemigroup of $\mathbb{N}_0$ whose complement $\mathbb{N}_0\backslash H$ in $\mathbb{N}_0$ is finite, which is called a *numerical semigroup*. The cardinality of $\mathbb{N}_0\backslash H$ is said to be the *genus* of H, which is denoted by $g(H)$. We say that H is *Weierstrass* if there exists a pointed curve (C, P) such that $H(P) = H$. Hurwitz' question in [3] was whether every numerical semigroup is Weierstrass. It had been a long-standing problem. Buchweitz [1] finally showed that not every numerical semigroup is Weierstrass. Namely, he gave a non-Weierstrass semigroup of genus 16. Using the Buchwitz' method we can show that for any $g \geqq 17$ there exists a non-Weierstrass semigroup of genus g (for example, see [8]). On the other hand, one of the authors proved that every numerical semigroup of genus $g \leqq 7$ (resp. every primitive numerical semigroup of genus $g = 8, 9$) is Weierstrass where a numerical semigroup H is said to be *primitive* if the largest integer not in H is less than twice the smallest positive integer in H (see [6], [9]).

In this paper we show that every non-primitive numerical semigroup of genus 8 is Weierstrass. In Section 2 using the known facts we show that any non-primitive n-semigroup of genus 8 is Weierstrass for $n \neq 6$ where a numerical semigroup H is called an *n-semigroup* if the minimum positive integer in H is n. In Section 3 for any non-primitive 6-semigroup H of genus 8 we construct a double covering of a curve with a ramification point P such that $H(P) = H$. Combining our result with Theorem 5.5 in [6] we see that every numerical semigroup of genus 8 is Weierstrass.

2 Non-primitive n-semigroups of genus 8 for $n \neq 6$

In this section we review the known facts and apply these results to our case. For a 2-semigroup H there exists a hyperelliptic curve C such that $H(P) = H$ for any Weierstrass point P on C. This result is classical. We know that every 3-semigroup is Weierstrass, which is due to Maclachlan [11]. Moreover, one of the authors proved that every 4-semigroup (resp. every 5-semigroup) is Weierstrass (see [4] (resp. [5])).

By the above notes it suffices to show that any non-primitive n-semigroup of genus 8 is Weierstrass for $n \geqq 6$. By the way there is only one non-primitive n-semigroup of genus 8 with $n \geqq 7$. The unique semigroup H_7 is generated by $7, 9, 10, 11, 12$ and 13. In view of [7] there is a cyclic covering of an elliptic curve of degree 8 which has only two ramification points P_1 and P_2, which are totally ramified, such that $H(P_1) = H(P_2) = H_7$.

3 Non-primitive 6-semigroups of genus 8

In this section we show that for any non-primitive 6-semigroup H of genus 8 there exists a double covering of a curve with a ramification point P such that $H(P) = H$. We denote by $M(H)$ the minimal set of generators for the semigroup H. The following table shows all non-primitive 6-semigroups of genus 8.

	$M(H)$	$\mathbb{N}_0 \backslash H$
(1)	$\{6, 7, 10, 11\}$	$\{1, 2, 3, 4, 5, 8, 9, 15\}$
(2)	$\{6, 8, 9, 10\}$	$\{1, 2, 3, 4, 5, 7, 11, 13\}$
(3)	$\{6, 8, 9, 11\}$	$\{1, 2, 3, 4, 5, 7, 10, 13\}$
(4)	$\{6, 8, 10, 11, 13\}$	$\{1, 2, 3, 4, 5, 7, 9, 15\}$
(5)	$\{6, 8, 10, 11, 15\}$	$\{1, 2, 3, 4, 5, 7, 9, 13\}$
(6)	$\{6, 9, 10, 11, 13\}$	$\{1, 2, 3, 4, 5, 7, 8, 14\}$
(7)	$\{6, 9, 10, 11, 14\}$	$\{1, 2, 3, 4, 5, 7, 8, 13\}$

Proposition 3.1. *Let H be one of the following 6-semigroups:* (2) $M(H) = \{6, 8, 9, 10\}$, (4) $M(H) = \{6, 8, 10, 11, 13\}$ *and* (5) $M(H) = \{6, 8, 10, 11, 15\}$. *Then there is a double covering of a curve of genus 2 with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = H$.*

Proof. Let C be a curve of genus 2. Take an ordinary point P on C. We want to construct a double covering of C with the ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = H$.

Case (2) $M(H) = \{6, 8, 9, 10\}$**.** We consider the divisor $D = 5P$. The degree of $2D - P$ is $9 > 4$, which implies that the divisor $2D - P$ is very ample. Hence we have

$$2D \sim P + (\text{some divisor}) = R$$

where R is a reduced divisor. Here for any two divisors D_1 and D_2 on C, $D_1 \sim D_2$ means that D_1 and D_2 are linearly equivalent. Let $\mathcal{L}$ be an invertible sheaf on C such that $\mathcal{L} \simeq \mathcal{O}_C(-D)$. Now we have isomorphisms

$$\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_C(-2D) \simeq \mathcal{O}_C(-R) \subset \mathcal{O}_C.$$

Using the composition of the above two isomorphisms we can construct a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R (See [12]). By Riemann-Hurwitz formula the genus of $\tilde{C}$ is 8. Let $\tilde{P} \in \tilde{C}$ be the ramification point of π over P. By Proposition 2.1 in [10] we obtain

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2n\tilde{P})) = h^0(C, \mathcal{O}_C(nP)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(nP))$$

for any positive integer n. Hence we get

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(8\tilde{P})) = h^0(C, \mathcal{O}_C(4P)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(4P)) = 3 \qquad \text{and}$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = h^0(C, \mathcal{O}_C(5P)) + h^0(C, \mathcal{L} \otimes \mathcal{O}_C(5P)) = 5,$$

which implies that $9 \in H(\tilde{P})$. Since $\tilde{P}$ is the ramification point over P with $M(H(P)) = \{3, 4, 5\}$, the semigroup $H(\tilde{P})$ contains 6, 8 and 10. In view of $g(H) = 8$ we must have $H(\tilde{P}) = H$.

Case (4) $M(H) = \{6, 8, 10, 11, 13\}$**.** Let Q be a unique point on C such that the divisor $P + Q$ is a canonical divisor K. Consider the divisor $D = 6P - Q$. Since the divisor $2D - P$ is very ample, we have

$$2D \sim P + (\text{some divisor}) = R$$

where R is a reduced divisor. In the same way as in the above we get a double covering

$$\pi : \tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = 4 + h^0(C, \mathcal{O}_C(-P + Q)) = 4,$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(12\tilde{P})) = 5 + h^0(C, \mathcal{O}_C(Q)) = 6 \qquad \text{and}$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(14\tilde{P})) = 6 + h^0(C, \mathcal{O}_C(P + Q)) = 8,$$

we see that $H(\tilde{P})$ contains 11 and 13. Hence we get $H(\tilde{P}) = H$.

Case (5) $M(H) = \{6, 8, 10, 11, 15\}$**.** Let Q be a point on C distinct from P such that the divisor $P + Q$ is not a canonical divisor K. Consider the divisor $D = 6P - Q$. We have

$$2D \sim P + (\text{some divisor}) = R$$

where R is a reduced divisor. In the same way as in the above we get a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = 4, h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(12\tilde{P})) = 6,$$
$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(14\tilde{P})) = 6 + h^0(C, \mathcal{O}_C(P+Q)) = 7 \qquad \text{and}$$
$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(16\tilde{P})) = 7 + h^0(C, \mathcal{O}_C(2P+Q)) = 9,$$

we see that $H(\tilde{P})$ contains 11 and 15. Hence we get $H(\tilde{P}) = H$. □

Remark.

1. All the seven remainder semigroups could be treated by Stöhr's methods as in [14].
2. The case (1) $M(H) = \{6, 7, 10, 11\}$ is a particular case of [16; Korollar 3]. See also [15; p. 204 and 208].
3. The case (2) $M(H) = \{6, 8, 9, 10\}$ is a particular case of [2; p. 422] by taking $n_1 = n_5 = 0$, $n_2 = 1$, $n_3 = 3$ and $n_4 = 2$.
4. The case (4) $M(H) = \{6, 8, 10, 11, 13\}$ and (5) $M(H) = \{6, 8, 10, 11, 15\}$ are particular cases of [13].

Proposition 3.2. *Let H be one of the following 6-semigroups:* (1) $M(H) = \{6, 7, 10, 11\}$, (3) $M(H) = \{6, 8, 9, 11\}$ *and* (7) $M(H) = \{6, 9, 10, 11, 14\}$. *Then there is a double covering of a non-hyperelliptic curve of genus 3 with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = H$.*

Proof. **Case** (1) $M(H) = \{6, 7, 10, 11\}$**.** Let C be a non-hyperelliptic curve of genus 3 with no point S such that $M(H(S)) = \{3, 4\}$. Let P be a Weierstrass point on C. Then we have $M(H(P)) = \{3, 5, 7\}$. Let Q be a unique point on C such that the divisor $3P + Q$ is a canonical divisor K. In this case Q is distinct from P. Consider the divisor $D = 4P - Q$. We want to show that

$$2D \sim P + (\text{some divisor}) = R$$

where R is a reduced divisor. It suffices to show that the linear system $|2D - P|$ is base-point free where for a divisor E on C the linear system $|E|$ means the set of effective divisors which are linearly equivalent to E. Assume that $|2D - P|$ were not base-point free. Then we get $2D - P \sim K + T$ for some point T. Hence we have $7P - 2Q \sim 3P + Q + T$, which implies that $4P \sim 3Q + T$. Since $|4P|$ is not base-point free, we should have $P = T$. Thus, we obtain $3P \sim 3Q$. Moreover, $K \sim 3P + Q \sim 4Q$, which implies that $M(H(Q)) = \{3, 4\}$. This is a contradiction. By Mumford's method we can construct a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R where $\mathcal{L} \simeq \mathcal{O}_C(-D)$. It suffices to show that 7 and 11 are contained in $H(\tilde{P})$ where $\tilde{P}$ is the ramification point over P. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(6\tilde{P})) = 2 + h^0(C, \mathcal{O}_C(-P + Q)) = 2 \quad \text{and}$$
$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(8\tilde{P})) = 2 + h^0(C, \mathcal{O}_C(Q)) = 3,$$

we see that $H(\tilde{P})$ contains 7. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = 3 + h^0(C, \mathcal{O}_C(P + Q)) = 4 \quad \text{and}$$
$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(12\tilde{P})) = 4 + h^0(C, \mathcal{O}_C(2P + Q)) = 6,$$

we see that $H(\tilde{P})$ contains 11.

Case (7) $M(H) = \{6, 9, 10, 11, 14\}$**.** Let C be a non-hyperelliptic curve of genus 3 with a Weierstrass point P satisfying $M(H(P)) = \{3, 5, 7\}$. Let A, B and U be distinct points on C different from P such that the divisor $P + A + B + U$ is linearly equivalent to a canonical divisor K. Consider the divisor $D = 5P - A - B$. We want to show that

$$2D \sim P + (\text{ some divisor }) = R$$

where R is a reduced divisor. Assume that $|2D - P|$ is not base-point free. Then we get $9P - 2A - 2B \sim K + S$ for some point S. If $S \neq P$, then we may assume that $K + S$ does not contain P, because K is base-point free. Hence we get our desired result. If $S = P$, then we replace B by U. Then we get $9P - 2A - 2U \sim K + S$ for some point S distinct from P, because if $S = P$ we get $U = B$, a contradiction. We can construct a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R where $\mathcal{L} \simeq \mathcal{O}_C(-D)$. It suffices to show that $H(\tilde{P}) \ni 9, 11$. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(8\tilde{P}) = 2 + h^0(C, \mathcal{O}_C(-P + A + B)) = 2 \qquad \text{and}$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = 3 + h^0(C, \mathcal{O}_C(A + B)) = 4,$$

we see that $H(\tilde{P})$ contains 9. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(12\tilde{P})) = 4 + h^0(C, \mathcal{O}_C(P + A + B)) = 6,$$

we see that $H(\tilde{P})$ contains 11.

Case (3) $M(H) = \{6, 8, 9, 11\}$**.** Let C be a non-hyperelliptic curve of genus 3 with a Weierstrass point P satisfying $M(H(P)) = \{3, 4\}$. We consider C as a canonical curve in $\mathbb{P}^2$. Let A and B be distinct points different from P such that $P + A + B + S$ is a canonical divisor for some point S. Then there is no line bitangent to A and B, because A and B are distinct from P. We set $D = 5P - A - B$. We want to show that

$$2D \sim P + (\text{ some divisor }) = R$$

where R is a reduced divisor. Assume that $2D - P$ were not base-point free. Then we get $2D - P \sim K + T$ for some point T. Hence we obtain $9P - 2A - 2B \sim 4P + T$, because $M(H(P)) = \{3, 4\}$ implies that $4P$ is a canonical divisor. Thus, we have $5P \sim 2A + 2B + T$. Since the divisor $5P$ has a base point, we must have $T = P$, because A and B are distinct from P. Hence, we get $K \sim 4P \sim 2A + 2B$. This contradicts the assumption that there is no line bitangent to A and B. We can construct a double covering

$$\pi : \tilde{C} = \mathrm{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$$

whose branch locus is R where $\mathcal{L} \simeq \mathcal{O}_C(-D)$. It suffices to show that $H(\tilde{P})$ contains 9 and 11. Since we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(8\tilde{P})) = 3 + h^0(C, \mathcal{O}_C(-P + A + B)) = 3 \qquad \text{and}$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(10\tilde{P})) = 3 + h^0(C, \mathcal{O}_C(A + B)) = 4,$$

we see that $H(\tilde{P})$ contains 9, because of $H(P) \not\ni 5$. Since $P + A + B \sim K - S$, we have

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(12\tilde{P})) = 4 + h^0(C, \mathcal{O}_C(P + A + B)) = 6.$$

Hence, $H(\tilde{P})$ contains 11. □

Proposition 3.3. *Let H be the 6-semigroup with* (6) $M(H) = \{6, 9, 10, 11, 13\}$. *Then there is a double covering of a curve of genus* 4 *with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = H$.*

Proof. Let E be an elliptic curve with the origin Q'. For a point P' of E we denote by $[P']$ the element corresponding to P' when we consider E as the abelian group with identity element $[Q']$. Moreover, for any integer n, $n[P']$ means n times $[P']$. Let P_1' be a point of E such that $P_1' \neq Q'$ and $2[P_1'] = [Q']$, i.e., $2P_1' \sim 2Q'$. Moreover, P_2' denotes a point of E such that

$$P_2' \neq Q',\ P_2' \neq P_1' \quad \text{and} \quad -5[P_1'] = 3[P_2'], \quad \text{i.e.,} \quad 5P_1' + 3P_2' \sim 8Q'.$$

Take $z \in k(E)$ such that $\operatorname{div}(z) = 5P_1' + 3P_2' - 8Q'$. Let $\pi : \tilde{C} \longrightarrow E$ be the surjective morphism corresponding to the inclusion $k(E) \subset k(E)(z^{1/8}) = k(\tilde{C})$. Let $y \in k(\tilde{C})$ and $\sigma \in \operatorname{Aut}(k(\tilde{C})/k(E))$ such that $\sigma(y) = \zeta_8 y$ and $\operatorname{div}_E(y^8) = 5P_1' + 3P_2' - 8Q'$, where ζ_8 is a primitive 8-th root of unity. Then there are only two ramification points $\tilde{P}_1$ and $\tilde{P}_2$ over P_1' and P_2' respectively and the ramification numbers are 8. Hence by the Riemann-Hurwitz relation the genus of $\tilde{C}$ is 8. We have

$$\operatorname{div}(y) = 5\tilde{P}_1 + 3\tilde{P}_2 - \pi^*(Q').$$

Since the divisor of dy is invariant under the action of σ, we have

$$\operatorname{div}(dy) = 4\tilde{P}_1 + 2\tilde{P}_2 - 2\pi^*(Q') + \sum_{i=1}^{3} \pi^*(R_i')$$

where R_i''s are points of E which are distinct from P_1', P_2' and Q'. For any $f \in k(E)$, we set $\operatorname{div}_E(f) = \sum_{P' \in E} n(P')P'$. Then for any $r \in \mathbb{N}_0$ we obtain

$$\begin{aligned}\operatorname{div}\left(\frac{f\,dy}{y^{1-r}}\right) &= \{8n(P_1') + 4 + 5(r-1)\}P_1 + \{8n(P_2') + 2 + 3(r-1)\}P_2 \\ &\quad + \{n(Q') - r - 1\}\pi^*(Q') + \sum_{i=1}^{3}\{n(R_i') + 1\}\pi^*(R_i') \\ &\quad + \sum_{P' \in S} n(P')\pi^*(P'),\end{aligned}$$

where S is the set of points $P' \in E$ except P'_1, P'_2, Q' and R'_i's. We set

$$D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^{3} R'_i, \qquad D'_1 = -2Q' + \sum_{i=1}^{3} R'_i,$$

$$D'_2 = -3Q' + P'_1 + \sum_{i=1}^{3} R'_i, \qquad D'_3 = -4Q' + P'_1 + P'_2 + \sum_{i=1}^{3} R'_i,$$

$$D'_4 = -5Q' + 2P'_1 + P'_2 + \sum_{i=1}^{3} R'_i, \quad D'_5 = -6Q' + 3P'_1 + P'_2 + \sum_{i=1}^{3} R'_i,$$

$$D'_6 = -7Q' + 3P'_1 + 2P'_2 + \sum_{i=1}^{3} R'_i, \quad D'_7 = -8Q' + 4P'_1 + 2P'_2 + \sum_{i=1}^{3} R'_i.$$

Then for each $r = 0, 1, \dots, 7$, $f \in L(D'_r)$ implies that $f dy/y^{1-r}$ is a regular 1-form on $\tilde{C}$ where

$$L(D'_r) = \left\{ f \in k(E) \mid \mathrm{div}_E(f) \geq -D'_r \right\}.$$

Since we have

$$\sigma\left(\frac{dy}{y}\right) = \frac{d\sigma y}{\sigma y} = \frac{d\zeta_8 y}{\zeta_8 y} = \frac{dy}{y},$$

the form dy/y is regarded as a 1-form on E. Hence there exists $f \in k(E)$ such that $f dy/y$ is regular. Then we must have

$$\mathrm{div}_E(f) = P'_1 + P'_2 + Q' - \sum_{i=1}^{3} R'_i, \text{ i.e., } l(D'_0) = 1$$

where for any divisor D we denote by $l(D)$ the dimension of the k-vector space $L(D)$, because

$$\begin{aligned} 0 \leqq \mathrm{div}_{\tilde{C}}\left(\frac{f dy}{y}\right) &= \mathrm{div}_{\tilde{C}}(f) + \mathrm{div}_{\tilde{C}}\left(\frac{dy}{y}\right) \\ &= \mathrm{div}_{\tilde{C}}(f) - \tilde{P}_1 - \tilde{P}_2 - \pi^*(Q') + \sum_{i=1}^{3} \pi^*(R'_i). \end{aligned}$$

Moreover, we have $l(D'_r) = 1$ for all $r = 1, 2, \dots, 7$, because of $\deg(D'_r) = 1$ for all $r = 1, 2, \dots, 7$. First we will show that $l(D'_1 - P'_1) = 0$. If $l(D'_1 - P'_1) > 0$, then we have

$$-2Q' + \sum_{i=1}^{3} R'_i - P'_1 \sim D'_1 - P'_1 \sim 0 \sim D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^{3} R'_i,$$

which implies that $P_2' \sim Q'$. This is a contradiction. If $l(D_2' - P_1') > 0$, then we have

$$-3Q' + \sum_{i=1}^{3} R_i' = D_2' - P_1' \sim D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that $P_1' + P_2' \sim 2Q' \sim 2P_1'$. This is a contradiction. If $l(D_3' - P_1') > 0$, then we have

$$-4Q' + P_2' + \sum_{i=1}^{3} R_i' = D_3' - P_1' \sim D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that $P_1' + 2P_2' \sim 3Q'$. Since we have

$$5P_1' + 3P_2' \sim 8Q' \sim 4Q' + 4P_1',$$

we obtain

$$P_1' + 2P_2' + Q' \sim 4Q' \sim P_1' + 3P_2',$$

which implies that $Q' \sim P_2'$. This is a contradiction. If $l(D_4' - P_1') > 0$, then we have

$$-5Q' + P_1' + P_2' + \sum_{i=1}^{3} R_i' = D_4' - P_1' \sim D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that

$$2P_1' + 2P_2' \sim 4Q' \sim P_1' + 3P_2'.$$

This is a contradiction. If $l(D_5' - P_1') > 0$, then we have

$$-6Q' + 2P_1' + P_2' + \sum_{i=1}^{3} R_i' = D_5' - P_1' \sim D_0' = -P_1' - P_2' - Q' + \sum_{i=1}^{3} R_i',$$

which implies that

$$3P_1' + 2P_2' \sim 5Q' \sim Q' + P_1' + 3P_2'.$$

Hence we have

$$2Q' \sim 2P_1' \sim Q' + P_2'.$$

This is a contradiction. Now we have

$$6Q' \sim 2Q' + P_1' + 3P_2' \sim 2P_1' + P_1' + 3P_2' = 3P_1' + 3P_2'.$$

Hence we get

$$D'_6 - P'_1 = -7Q' + 2P'_1 + 2P'_2 + \sum_{i=1}^{3} R'_i \sim -Q' - P'_1 - P'_2 + \sum_{i=1}^{3} R'_i = D'_0 \sim 0,$$

which implies that $l(D'_6) = l(D'_6 - P'_1) = 1$. If $l(D'_7 - P'_1) > 0$, then we have

$$-8Q' + 3P'_1 + 2P'_2 + \sum_{i=1}^{3} R'_i = D'_7 - P'_1 \sim D'_0 = -P'_1 - P'_2 - Q' + \sum_{i=1}^{3} R'_i,$$

which implies that

$$4P'_1 + 3P'_2 \sim 7Q' \sim 4P'_1 + 3Q'.$$

Hence we get

$$3P'_2 + Q' \sim 4Q' \sim P'_1 + 3P'_2.$$

This is a contradiction. By the above we have

$$l(D'_i - P'_1) = 0 \text{ for all } i \neq 6 \text{ and } l(D'_6 - P'_1) = 1, l(D'_6 - 2P'_1) = 0.$$

For each $r = 0, 1, \cdots, 7$ we take a non-zero element $f_r \in L(D'_r)$ and we set $\phi_r = f_r dy/y^{1-r}$. Then by the above we see the following:

$$\begin{aligned}
&\mathrm{ord}_{\tilde{P}_1}(\phi_0) = 8 - 1 = 7 = 8 - 1, && \mathrm{ord}_{\tilde{P}_1}(\phi_1) = 0 + 4 = 5 - 1,\\
&\mathrm{ord}_{\tilde{P}_1}(\phi_2) = -8 + 9 = 1 = 2 - 1, && \mathrm{ord}_{\tilde{P}_1}(\phi_3) = -8 + 14 = 6 = 7 - 1,\\
&\mathrm{ord}_{\tilde{P}_1}(\phi_4) = -16 + 19 = 3 = 4 - 1, && \mathrm{ord}_{\tilde{P}_1}(\phi_5) = -24 + 24 = 0 = 1 - 1,\\
&\mathrm{ord}_{\tilde{P}_1}(\phi_6) = -16 + 29 = 13 = 14 - 1, && \mathrm{ord}_{\tilde{P}_1}(\phi_7) = -32 + 34 = 2 = 3 - 1.
\end{aligned}$$

We note that $n \in \mathbb{N}_0 \backslash H(\tilde{P}_1)$ if and only if there exists a regular 1-form ϕ on $\tilde{C}$ such that $\mathrm{ord}_{\tilde{P}_1}(\phi) = n - 1$. Hence we obtain

$$\mathbb{N}_0 \backslash H(\tilde{P}_1) = \{1, 2, 3, 4, 5, 7, 8, 14\}.$$

Let K be the subfield of $k(\tilde{C})$ consisting of the elements which are fixed by the automorphism σ^4. We denote by C the curve with function field K. Let $\eta : C \longrightarrow E$ be the covering corresponding to the inclusion $k(E) \subset K$. Then η is a morphism of degree 4 with only two ramification point P_i over P'_i for $i = 1, 2$. Hence, the genus of C is 4. Moreover, the canonical morphism $\tilde{C} \longrightarrow C$ is a double covering with the ramification point $\tilde{P}_1$ over P_1 satisfying $M(H(\tilde{P}_1)) = \{6, 9, 10, 11, 13\}$. □

The following theorem can be deduced from Propositions 3.1, 3.2 and 3.3.

Theorem 3.4. *Every non-primitive* 6*-semigroup of genus* 8 *is Weierstrass.*

Combining the statement in Section 2 and Theorem 3.4 with Theorem 5.5 in [6] we get the following result:

Corollary 3.5. *Any numerical semigroup of genus* 8 *is Weierstrass.*

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