Weierstrass semigroups whose minimum positive integers are even

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Abstract. We consider three subsets of the set of $2n$-semigroups where for a positive integer n a $2n$-semigroup means a numerical semigroup whose minimum positive integer is $2n$. These three subsets are obtaind by the Weierstrass semigroups of total ramification points on a cyclic covering of the projective line, the Weierstrass semigroups of ramification points on a double covering of a non-singular curve and the Weierstrass semigroups of points on a non-singular curve. We show that three subsets are different for $n \geqq 3$.

1 Introduction

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ of H in $\mathbb{N}_0$ is finite. The cardinality of the set $\mathbb{N}_0 \backslash H$ is said to be the *genus* of H which is denoted by $g(H)$. Let C be a complete non-singular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. We denote by $k(C)$ the field of rational functions on C. For any point P of C we set

$$H(P) = \{n \in \mathbb{N}_0 | \text{ there exists } f \in k(C) \text{ such that } (f)_\infty = nP\},$$

which is called the *Weierstrass semigroup* of the point P. A numerical semigroup is said to be *Weierstrass* if there exists a pointed curve (C, P) such that $H = H(P)$. Let m be a positive integer. An m-semigroup means a numerical

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semigroup whose minimum positive integer is m. For any positive integer n we are interested in three subsets of the set of the $2n$-semigroups. The first one is obtained by the Weierstrass semigroups of total ramification points of a cyclic covering of the projective line $\mathbb{P}^1$. The second subset consists of the Weierstrass semigroups of ramification points of a double covering of a curve. The elements of the last subset are the Weierstrass $2n$-semigroups. In this paper we show that for any $n \geqq 3$ the three subsets are different. In fact, we construct a ramification point of a double covering of a curve whose Weierstrass semigroup is not attained by that of a total ramification point of a cyclic covering of the projective line in Section 2. In Section 3 we give a Weierstrass $2n$-semigroup which is not that of any ramification point of a double covering of a curve.

2 $2n$-semigroups of double covering type

Let m be an integer larger than 1. An m-semigroup is said to be *cyclic* if there is a total ramification point P on a cyclic covering of the projective line $\mathbb{P}^1$ with degree m such that $H(P) = H$. For positive integers $a_1, a_2, \ldots, a_l$ we denote by $\langle a_1, a_2, \ldots, a_l \rangle$ the semigroup generated by $a_1, a_2, \ldots, a_l$.

Example 2.1. Let q be an integer larger than m which is relatively prime to m. Then the m-semigroup $\langle m, q \rangle$ is cyclic.

Proof. Let C be a curve whose function field $k(C) = k(x, y)$ is defined by an equation of the form $y^m = \prod_{i=1}^{q}(x - c_i)$ where c_i's are distinct elements of k. Let $f : C \longrightarrow \mathbb{P}^1$ be the morphism of curves sending any point P of C to the projective coordinate $(1 : x(P))$. Then f is a cyclic covering of degree m. We denote by P_∞ the point of C with $f(P_\infty) = (0 : 1)$. Then P_∞ is a total ramification point of f whose Weierstrass semigroup is $\langle m, q \rangle$ (see [1], Theorem 2.1 for any prime m and in the case where m is not prime the same method also works well). □

It is easy to check the following:

Remark 2.2. Every m-semigroup is cyclic for $m = 2, 3$.

But not every 4-semigroup is cyclic. In fact, for an m-semigroup H we set $S(H) = \{m, s_1, \ldots, s_{m-1}\}$ where $s_i = \mathrm{Min}\{h \in H | h \equiv i(m)\}$ for

$i = 1, \ldots, m-1$, which is called the *standard basis* for H. Observe that if $s_i = a_i m + i$ for $i = 1, \ldots, m-1$ then $g(H) = \sum_{i=1}^{m-1} a_i$. We have the following result.

Remark 2.3. A 4-semigroup is cyclic if and only if we have $g(H) \geqq 3\left[\frac{s_2}{4}\right]$ ([2], Lemmas 4.1 and 4.3).

In this paper we are interested in non-cyclic $2n$-semigroups which are Weierstrass. A $2n$-semigroup H is said to be *of double covering type* if there exists a ramification point P on a double covering of a curve such that $H(P) = H$. We give a non-cyclic $2n$-semigroup of double covering type for any $n \geqq 3$ in this section. We will use the following facts to construct such a semigroup.

Remark 2.4. Let H be an n-semigroup and l an odd number larger than $2c(H) - 2$ where $c(H) = \mathrm{Min}\{c \in \mathbb{N}_0 | c + H \subseteqq H\}$. We set $H_l = 2H + l\mathbb{N}_0$. Assume that $l \neq 2n - 1$. Then H_l is a $2n$-semigroup of genus $2g(H) + \frac{l-1}{2}$ with $S(H_l) = \{2n, 2s_1, \ldots, 2s_{n-1}, l, l + 2s_1, \ldots, l + 2s_{n-1}\}$ ([3], Lemma 2.1).

Remark 2.5. Let H, l and H_l be as in Remark 2.4. Assume that H is Weierstrass, i.e., we have a pointed curve (C, P) such that $H(P) = H$. Then there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = H_l$ ([3], Theorem 2.2).

Example 2.6. Let n be an odd number larger than 2. We set $H = \langle n, n+1, n+n-1 \rangle$. Let l be an odd number larger than $2c(H) - 2$ with $l \neq 2n - 1$. Then $H_l = 2H + l\mathbb{N}_0$ is a non-cyclic $2n$-semigroup of double covering type.

Proof. Let $S(H) = \{n, s_1, \ldots, s_{n-1}\}$ and $S(H_l) = \{2n, t_1, \ldots, t_{2n-1}\}$. Then by Remark 2.4 we have

$$t_2 = 2s_1 = 2(n+1),\ t_4 = 2s_2 = 4n + 4,$$

$t_{2n-4} = 2s_{n-2} = 2(n+n-1+n+n-1) = 8n-4$ and $t_{2n-2} = 2s_{n-1} = 4n-2$. Hence we get $t_2 + t_{2n-2} = 6n \neq 12n = t_4 + t_{2n-4}$. In view of $(2, 2n-2) = (4, 2n-4) = 2$ the $2n$-semigroup H_l is not cyclic ([4], Lemma 4.8). □

Example 2.7. Let n be an even number larger than 3. We set $H = \langle n, n+1, n+n-1 \rangle$. Let $l = 2qn + 1 \geqq 2c(H) - 1$ for some integer q. Then $H_l = 2H + l\mathbb{N}_0$ is a non-cyclic $2n$-semigroup of double covering type.

Proof. Let $S(H) = \{n, s_1, \ldots, s_{n-1}\}$ and $S(H_l) = \{2n, t_1, \ldots, t_{2n-1}\}$. Then we have

$$t_1 = 2qn + 1,\ t_{n-1} = 2qn + 1 + 2s_{\frac{n}{2}-1},$$

$$t_{n+1} = 2qn + 1 + 2s_{\frac{n}{2}} \text{ and } t_{2n-1} = 2qn + 1 + 2s_{n-1}.$$

Since n is even, we get $(1, 2n-1) = (n-1, n+1) = 1$. Assume that H_l is cyclic. Then we must have $t_1 + t_{2n-1} = t_{n-1} + t_{n+1}$ ([4], Lemma 4.8), which implies that

$$2n - 1 = s_{n-1} = s_{\frac{n}{2}-1} + s_{\frac{n}{2}} \geqq n + \frac{n}{2} - 1 + 2n + \frac{n}{2} = 4n - 1.$$

This is a contradiction. □

By Examples 2.6 and 2.7 we get the following result:

Theorem 2.8. *Let n be an integer larger than 2. Then the set of the $2n$-semigroups of double covering type contains properly the set of the cyclic $2n$-semigroups.*

3 Weierstrass $2n$-semigroups

First, using Hurwitz' formula we give a property of the standard basis for a $2n$-semigroup of double covering type.

Lemma 3.1. *Let n be an integer larger than 1 and $\tilde{H}$ a $2n$-semigroup of double covering type with $S(\tilde{H}) = \{2n, t_1, \ldots, t_{2n-1}\}$ where $t_i \equiv i$ mod $2n$. Then we have*

$$\sum_{k=1}^{n} \left[\frac{t_{2k-1}}{2n}\right] \geqq \sum_{i=1}^{n-1} \left[\frac{t_{2i}}{2n}\right].$$

Proof. There exists a double covering $\pi : \tilde{C} \longrightarrow C$ of curves with ramification point $\tilde{P}$ over a point P such that $H(\tilde{P}) = \tilde{H}$. Then we have

$$S(H(P)) = \left\{n, \frac{t_2}{2}, \frac{t_4}{2}, \ldots, \frac{t_{2n-2}}{2}\right\}.$$

By Hurwitz' formula we get

$$2g(\tilde{C})-2=2(2g(C)-2)+\sharp\{\text{ ramification points of }\pi\}>2(2g(C)-2)$$

where for a curve X we denote the genus of X by $g(X)$. Hence, we obtain $g(\tilde{C}) \geqq 2g(C)$. We also have

$$g(\tilde{C})=g(\tilde{H})=\sum_{j=1}^{2n-1}\left[\frac{t_j}{2n}\right]=\sum_{k=1}^{n}\left[\frac{t_{2k-1}}{2n}\right]+\sum_{i=1}^{n-1}\left[\frac{t_{2i}}{2n}\right]$$

$$=\sum_{k=1}^{n}\left[\frac{t_{2k-1}}{2n}\right]+g(H(P))=\sum_{k=1}^{n}\left[\frac{t_{2k-1}}{2n}\right]+g(C).$$

Therefore, we get the desired inequality. □

Using the criterion in Lemma 3.1 we get a $2n$-semigroup which is not of double covering type.

Example 3.2. For any integer $n \geqq 3$ let $\tilde{H}$ be a $2n$-semigroup with

$$S(\tilde{H})=\{2n\}\cup\{2n+2k-1 \mid k=1,\dots,n\}\cup\{4n+2i \mid i=1,\dots,n-1\}.$$

Then $\tilde{H}$ is not of double covering type.

Proof. For any $i\in\{1,\dots,2n-1\}$ let $t_i\in S(\tilde{H})$ with $t_i\equiv i \bmod 2n$. Assume that $\tilde{H}$ is of double covering type. By Lemma 3.1 we get

$$0\leqq\sum_{k=1}^{n}\left[\frac{t_{2k-1}}{2n}\right]-\sum_{i=1}^{n-1}\left[\frac{t_{2i}}{2n}\right]$$

$$=\sum_{k=1}^{n}\left[\frac{2n+2k-1}{2n}\right]-\sum_{i=1}^{n-1}\left[\frac{4n+2i}{2n}\right]=n-2(n-1)=-n+2\leqq-1,$$

which is a contradiction. □

We shall determine a unique minimal set of generators for $\tilde{H}$.

Remark 3.3. Let $\tilde{H}$ be as in Example 3.2. Then

$$M(\tilde{H})=\{2n\}\cup\{2n+2k-1|k=1,\dots n\}$$

is the minimal set of generators for the semigroup $\tilde{H}$.

Proof. Let i be an odd number with $1 \leqq i \leqq n-1$. Then we have

$$4n + 2i = 2(2n + i) \text{ with } i = 2 \cdot \frac{i+1}{2} - 1.$$

If i is an even number with $2 \leqq i \leqq n-1$, then $4n + 2i = (2n + i - 1) + (2n + i + 1)$. □

Theorem 3.4. *The $2n$-semigroup $\tilde{H}$ with minimal set*

$$M(\tilde{H}) = \{2n\} \cup \{2n + 2k - 1 | k = 1, \dots n\}$$

of generators is Weierstrass.

Proof. We set $a_0 = 2n$ and $a_k = 2n + 2k - 1$ for $k = 1, \dots, n$. Then we have $\alpha_0 = \alpha_1 = 3$ and $\alpha_k = 2$ for $2 \leqq k \leqq n$ where

$$\alpha_i = \mathrm{Min}\{\alpha \in \mathbb{N}_0 > 0 \mid \alpha a_i \in \langle a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle\}.$$

In fact, we obtain

$$3a_0 = a_1 + a_n,\ 2a_1 + a_i = 2a_0 + a_{i+1} \text{ for } 1 \leqq i \leqq n-1,$$

$$a_i + a_j = a_{i-1} + a_{j+1} \text{ for } 2 \leqq i \leqq j \leqq n-1,$$

$$a_i + a_n = a_0 + a_1 + a_{i-1} \text{ for } 2 \leqq i \leqq n.$$

Let $\varphi : k[X_0, X_1, \dots, X_n] \longrightarrow k[t^h]_{h \in \tilde{H}}$ be the k-algebra homomorphism sending X_i to t^{a_i} for each i. We denote by I the kernel of the morphism φ. We want to show that the polynomials corresponding to the above relations form a system of generators for the ideal I, that is to say,

$$X_0^3 - X_1 X_n \tag{1}$$

$$X_1^2 X_i - X_0^2 X_{i+1} \text{ for } 1 \leqq i \leqq n-1 \tag{2}$$

$$X_i X_j - X_{i-1} X_{j+1} \text{ for } 2 \leqq i \leqq j \leqq n-1 \tag{3}$$

$$X_i X_n - X_0 X_1 X_{i-1} \text{ for } 2 \leqq i \leqq n \tag{4}$$

generate the ideal I. In fact, let J be the ideal generated by the above polynomials. It suffices to show that any polynomial $f = \prod_{i=0}^{n} X_i^{\nu_i} - \prod_{i=0}^{n} X_i^{\mu_i} \in$

I with $\nu_i\mu_i = 0$ for any i is contained in J. To prove this we will consider the following six cases.

Case 1. $X_0X_1 \Big| \prod_{i=0}^{n} X_i^{\nu_i}$, i.e., $\prod_{i=0}^{n} X_i^{\nu_i}$ is divisible by X_0X_1. Then $X_kX_l \big| \prod X_i^{\mu_i}$ for some k and l with $2 \leqq k \leqq l \leqq n$. Using (3) and (4) we may decrease the degree of f.

Case 2. $X_0X_n \Big| \prod_{i=0}^{n} X_i^{\nu_i}$. Then $X_kX_l \big| \prod X_i^{\mu_i}$ for some k and l with $1 \leqq k \leqq l \leqq n-1$. If $k = 1$, using the inequality $a_1 + a_l < a_0 + a_n$ and (3)we may decrease the degree of f or assume that $X_1^2X_l \big| \prod X_i^{\mu_i}$, in which case using (2) we may decrease the degree of f. If $k \geqq 2$, using (3) and the above argument we may decrease the degree of f.

Case 3. $X_0 \Big| \prod_{i=0}^{n} X_i^{\nu_i}$. Since $a_0 + a_m$ with $1 \leqq m$ is odd and $a_p + a_q$ with $1 \leqq p \leqq q$ is even, we must have $X_0X_kX_l \big| \prod X_i^{\nu_i}$ with $0 \leqq k \leqq l$. If $k = 0$, then by (1) and (2) we may assume that $X_1 \big| \prod X_i^{\nu_i}$ and $X_pX_q \big| \prod X_i^{\mu_i}$ for some p and q with $2 \leqq p \leqq q \leqq n$. By (3) and (4) we may decrease the degree of f. If $k = 1$, then this case is reduced to Case 1. If $k \geqq 2$, by (3) this case is reduced to either Case 1 or Case 2.

Case 4. $X_1 \Big| \prod_{i=0}^{n} X_i^{\nu_i}$. Then we may assume that $X_1X_k \big| \prod X_i^{\nu_i}$ with $1 \leqq k \leqq n$ and $X_lX_m \big| \prod X_i^{\mu_i}$ with $2 \leqq l \leqq m \leqq n$. By (3) we may assume that $m = n$. Using (4) we may decrease the degree of f.

Case 5. $X_n \Big| \prod_{i=0}^{n} X_i^{\nu_i}$. Then we may assume that $X_lX_m \big| \prod X_i^{\mu_i}$ with $2 \leqq l \leqq m \leqq n-1$. Using (3) we may reduce this case to Case 4.

Case 6. $X_kX_l \Big| \prod_{i=0}^{n} X_i^{\nu_i}$ with $2 \leqq k \leqq l \leqq n-1$. By (3) we may reduce this case to either Case 4 or Case 5.

Hence we get $I = J$, i.e., the polynomials of type (1), (2), (3) or (4) generate the ideal I. We rewrite the relations among $a_0, a_1, \ldots, a_n$ as follows:

$$(\alpha_{10} + \alpha_{n0})a_0 = \alpha_{01}a_1 + \alpha_{0n}a_n \tag{5}$$

$$(\alpha_{01} + \alpha_{n1})a_1 + \alpha_{i+1\ i}\ a_i = \alpha_{10}a_0 + \alpha_{i\ i+1}\ a_{i+1} \text{ for } 1 \leqq i \leqq n-1 \tag{6}$$

$$\alpha_{i-1\ i}\ a_i + \alpha_{j+1\ j}\ a_j = \alpha_{i\ i-1}\ a_{i-1} + \alpha_{j\ j+1}\ a_{j+1} \text{ for } 2 \leqq i \leqq j \leqq n-1 \tag{7}$$

$$\alpha_{i-1\ i}\ a_i + \alpha_{0n} a_n = \alpha_{n0} a_0 + \alpha_{n1} a_1 + \alpha_{i\ i-1}\ a_{i-1} \text{ for } 2 \leqq i \leqq n \tag{8}$$

where we set $\alpha_{10} = 2$ and $\alpha_{ij} = 1$ if $(i, j) \neq (1, 0)$. We set $\mathbf{b}_i = \mathbf{e}_i \in \mathbb{Z}^{n+3}$ for $1 \leqq i \leqq n+3$ where $\mathbf{e}_i$ is the vector whose i-th component is 1 and other components are 0. We associate $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$ and $\mathbf{b}_{k+4}$ with $1 \leqq k \leqq n-1$ to $\alpha_{10}a_0$, $\alpha_{n0}a_0$, $\alpha_{01}a_1$, $\alpha_{n1}a_1$ and $\alpha_{k+1\ k}\ a_k$ with $1 \leqq k \leqq n-1$ respectively. Using (5), (6) with $i = 1$ and (7) with $i = j = k$, we can associate

$$\mathbf{b}_{n+4} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 \text{ and } \mathbf{b}_{n+k+4} = -\mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_{k+4} \text{ with } 1 \leqq k \leqq n-1$$

to $\alpha_{0n}a_n$ and $\alpha_{k\ k+1}\ a_{k+1}$ with $1 \leqq k \leqq n-1$ respectively. Let S be the subsemigroup of $\mathbb{Z}^{n+3}$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_{2n+3}$. We consider the semigroup algebra $k[S] = k[T^s]_{s \in S}$. Then we have a commutative diagram

$$\begin{array}{ccc} k[Y_1, \ldots, Y_{2n+3}] & \xrightarrow{\pi} & k[S] \\ \downarrow^{\eta} & & \downarrow^{\zeta} \\ k[X_0, \ldots, X_n] & \xrightarrow{\varphi} & k[\tilde{H}] \end{array}$$

where π sends Y_i to $T^{\mathbf{b}_i}$ and ζ maps $T^{\mathbf{b}_i}$ to t^{h_i} where $\mathbf{b}_i$ is associated to the element h_i of $\tilde{H}$. Moreover, η sends Y_i to g_i where g_i's $(i = 1, \ldots, 2n+3)$ in $k[X_0, \ldots, X_n]$ are the monomials derived from $\alpha_{10}a_0$, $\alpha_{n0}a_0$, $\alpha_{01}a_1$, $\alpha_{n1}a_1$ and $\alpha_{k+1\ k}\ a_k$ with $1 \leqq k \leqq n-1$, $\alpha_{0n}a_n$ and $\alpha_{k\ k+1}\ a_{k+1}$ with $1 \leqq k \leqq n-1$ respectively, where the monomial derived from βa_i is X_i^{β}. Hence, we see that the ideal $I = \text{Ker } \varphi$ is generated by the elements of the set $\eta(\text{Ker } \pi)$.

Lastly, we want to show that Spec $k[S]$ is an affine toric variety. It suffices to show that the semigroup algebra $k[S]$ is normal, i.e., $\sum_{i=1}^{2n+3} \mathbb{R}_+ \mathbf{b}_i \cap \mathbb{Z}^{n+3} \subseteqq S$ where $\mathbb{R}_+$ is the set of non-negative real numbers. Let $\mu \in \sum_{i=1}^{2n+3} \mathbb{R}_+ \mathbf{b}_i \cap \mathbb{Z}^{n+3}, = \sum_{i=1}^{2n+3} \lambda_i \mathbf{b}_i$. We may assume that $0 \leqq \lambda_i < 1$ for all i. We set $\mu = (\mu_1, \ldots, \mu_{n+3})$. Then we have

$$\mu_1 = \lambda_1 + \lambda_{n+4} - \sum_{k=5}^{n+3} \lambda_{n+k},\ \mu_2 = \lambda_2 + \lambda_{n+4},\ \mu_3 = \lambda_3 - \lambda_{n+4} + \sum_{k=5}^{n+3} \lambda_{n+k},$$

$$\mu_4 = \lambda_4 + \sum_{k=5}^{n+3} \lambda_{n+k},\ \mu_k = \lambda_k + \lambda_{n+k} \text{ for } 5 \leqq k \leqq n+3.$$

Since μ_1 is an integer and $0 \leqq \lambda_i < 1$, we get $\mu_1 \geqq -(n+3-4)+1 = -n+2$. If $\mu_1 \geqq 0$, then $\mu \in S$, because $\mu_j \geqq 0$ for all j. We set $\mu_1 = -l$ with $1 \leqq l \leqq n-2$. Then we have

$$\mu_3 = \lambda_1 + \lambda_3 + l \geqq l \text{ and } \mu_4 = \lambda_4 + \lambda_1 + \lambda_{n+4} + l \geqq l.$$

Since at least $l+1$ elements among $\lambda_{n+5}, \ldots, \lambda_{2n+3}$ are non-zero, at least $l+1$ elements among $\mu_5, \ldots, \mu_{n+3}$ should be 1. Hence we get $\mu \in S$, which implies that Spec $k[S]$ is an affine toric variety. By [2], Corollary 4.9, $\tilde{H}$ is Weierstrass. □

By Example 3.2 and Theorem 3.4 we get the following result:

Corollary 3.5. *Let n be an integer larger than 2. Then the set of the Weierstrass $2n$-semigroups contains properly the set of the $2n$-semigroups of double covering type.*

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