Double coverings of curves and non-Weierstrass semigroups

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We give a new method of constructing a non-Weierstrass semigroup H, which means that there is no smooth projective pointed curve over an algebracally closed field of characteristic 0 whose Weierstrass semigroup is H. This method depends on a description of a pointed smooth projective curve such that there exists a double covering of the curve ramified over the point with a certain condition on the genus of the covering curve. Using this we find non-Weierstrass semigroups whose minimum positive integers are 8 and 12 respectively.

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1 Introduction

Let $\mathbb{N}_0$ be the additive semigroup of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0\backslash H$ is a finite set, whose cardinality is called the *genus* of H, denoted by $g(H)$. For a positive integer m an *m-semigroup* means a numerical semigroup whose minimum positive integer is m. In this paper a *curve* means a complete smooth 1-dimensional algebraic variety over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{n \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP\},$$

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which is called the *Weierstrass semigroup* of P where $k(C)$ denotes the field of rational functions on C and $(f)_\infty$ is the pole divisor of a rational function f. For distinct two points P and Q of C we set

$$H(P,Q) = \{(n,l) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = nP + lQ\},$$

which is called the *Weierstrass semigroup of a pair of points* P and Q. Then $H(P)$ is a numerical semigroup of genus g where g is the genus of the curve. A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C,P) such that $H = H(P)$. We know that for any positive integer $m \leqq 5$ every m-semigroup is Weierstrass (see [11], [6] and [7] in the cases $m = 3$, 4 and 5 respectively). But it is showed that for any integer $m \geqq 13$ not every m-semigroup is Weierstrass (see [1] and [8]). Therefore we have an open problem as follows:

For each $6 \leqq m \leqq 12$ *is every* m*-semigroup Weiersrass ?*

The aim of this paper is to show the following:

Main Theorem. i) Let $l \geqq 2$ and n an odd number with $n \geqq 16l + 19$ (resp. $16l + 27$). Then

$$\langle 8, 12, 8l+2, 8l+6, n, n+4\rangle \text{ (resp. } \langle 8, 12, 8l+6, 8l+10, n, n+4\rangle).$$

is a non-Weierstrass 8-semigroup.
ii) *Let* $l \geqq 2$ *and* n *an odd number with* $n \geqq 24l + 27$*. Then*

$$\langle 12, 16, 20, 12l+2, 12l+6, 12l+10, n, n+4\rangle$$

is a non-Weierstrass 12*-semigroup.*

In Main Theorem we use the following notation: For positive integers $a_1, a_2, \dots, a_m$ we denote by $\langle a_1, a_2, \dots, a_m\rangle$ the additive monoid generated by $a_1, a_2, \dots, a_m$. To prove Main Theorem, i.e., to find non-Weierstrass 8-semigroups and 12-semigroups we give a new method of constructing non-Weierstrass semigroups. This method is different from those of Buchweitz [1] and Torres [12]. It depends on a characterization of a double covering of a curve satisfying a certain relation between the genera of the base curve and the covering curve in terms of Weierstrass semigroups of pairs of points. To explain the characterization we need some terminologies. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$ where $c(H) = \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subsetneqq H\}$. For an m-semigroup H the set $S(H) = \{m, s_1, \dots, s_{m-1}\}$ denotes the standard basis for H where $s_i = \min\{h \in H \mid h \equiv i \bmod m\}$ for each i with $1 \leqq i \leqq m-1$. We set $s_{\max} = \max\{s_1, s_2, \dots, s_{m-1}\}$. For a numerical semigroup $\tilde{H}$ we denote by $d_2(\tilde{H})$ the set consisting of $\dfrac{\tilde{h}}{2}$ with even $\tilde{h} \in \tilde{H}$, which becomes a numerical semigroup.

If $\pi : \tilde{C} \longrightarrow C$ is a double covering of a curve with a ramification point $\tilde{P}$, then $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$ (For example, see [12]). A numerical semigroup $\tilde{H}$ is called the *double covering type* if there exists a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ such that $\tilde{H} = H(\tilde{P})$.

In [12] Torres proved the following:

If $\tilde{H}$ is a Weierstrass semigroup with $g(\tilde{H}) \geqq 6g(d_2(\tilde{H})) + 4$, then it is the double covering type, hence $d_2(\tilde{H})$ is Weierstrass.

We are interested in the converse of the Torres' result as follows:

Problem A. *Let $\tilde{H}$ be a numerical semigroup with $g(\tilde{H}) \geqq 6g(d_2(\tilde{H})) + 4$. Assume that $d_2(\tilde{H})$ is Weierstrass. Then is $\tilde{H}$ Weierstrass ? That is to say, is $\tilde{H}$ the double covering type ?*

We solve Problem A negatively by giving non-Weierstrass 8-semigroups $\tilde{H}$ with $g(\tilde{H}) \geqq 6g(d_2(\tilde{H})) + 4$ and Weierstrass semigroups $d_2(\tilde{H})$. We get such 8-semigroups using the following characterization of numerical semigroups of double covering type with a certain condition.

Theorem B. *Assume that an m-semigroup H is non-symmetric. Let $s_i \neq s_{\max}$ with no j such that $s_j = s_i + s_k$ for some k. Then the following are equivalent:*
i) *There exists an odd integer $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$ such that $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_i - m))\mathbb{N}_0$ is the double covering type.*
ii) *For any odd $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$ the semigroup $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_i - m))\mathbb{N}_0$ is the double covering type.*
iii) *There exists a pointed curve (C, P) with $H(P) = H$ and a point $P_1 \neq P$ such that $(s_i - m, 1) \in H(P, P_1)$.*

In Section 2 we study about numerical semigroups $\tilde{H}$ satisfying $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - 1$ where we set $H = d_2(\tilde{H})$ and n is as in Theorem B. Especially we prove the same statements as those of Theorem B in the case of $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0$. In Section 3 we consider the case where $d_2(\tilde{H})$ is a symmetric semigroup satisfying $g(\tilde{H}) = 2g(d_2(\tilde{H})) + \dfrac{n-1}{2} - 1$. In this case we show a result similar to our previous result ([10]) on Weierstrass semigroups $\tilde{H}$ of double covering type satisfying $g(\tilde{H}) = 2g(d_2(\tilde{H})) + \dfrac{n-1}{2}$. In Section 4 we prove Theorem B. Using Theorem B we solve Problem A negatively in Section 5. Moreover, we show Main Theorem. Namely non-Weierstrass 8-semigroups and non-Weierstrass 12-semigroups are given.

2 Semigroups of double covering type

In [10] we have the following result:

Remark 2.1 *Let H be an m-semigroup and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. We set $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h}$ is odd $\}$. Assume that $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$. Then the following are equivalent:*
i) $g(\tilde{H}) = 2g(H) + \frac{n-1}{2}$,
ii) $\tilde{H} = 2H + n\mathbb{N}_0$.
In this case, if H is Weierstrass, then $\tilde{H}$ is the double covering type.

We also have a description similar to Remark 2.1 in the case $g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - 1$ except the statement related to the double covering.

Lemma 2.2 *Let H, $\tilde{H}$ and n be as in Remark 2.1. Then the following are equivalent:*
i) $g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - 1$,
ii) *There exists $i \in \{1, 2, \ldots, m-1\}$ with no j satisfying $s_j = s_i + s_k$ for some k such that*

$$\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_i - m))\mathbb{N}_0.$$

Proof. By [10] we have

$$S(2H + n\mathbb{N}_0) = \{2m, 2s_1, \ldots, 2s_{m-1}, n, n + 2s_1, \ldots, n + 2s_{m-1}\}.$$

First, we prove that i) implies ii). By Remark 2.1 and the assumption i) we obtain

$$\mathbb{N}_0 \backslash \tilde{H} = (\mathbb{N}_0 \backslash (2H + n\mathbb{N}_0)) \backslash \{n + 2(s_i - m)\}$$

for some i. Assume that there were j such that $s_j = s_i + s_k$ for some k. Then $n + 2s_j - 2m = 2s_k + n + 2(s_i - m) \in \tilde{H}$, which implies that $g(\tilde{H}) \leqq 2g(H) + \frac{n-1}{2} - 2$. This is a contradiction.

Conversely, we assume that $g(\tilde{H}) < 2g(H) + \frac{n-1}{2} - 1$. Then there exists j distinct from i such that $n + 2s_j - 2m \in \tilde{H}$. Hence

$$n + 2s_j - 2m = c_0 \cdot 2m + \sum_{l=1}^{m-1} c_l \cdot 2s_l + d_0 \cdot n + \sum_{l=1}^{m-1} d_l(n + 2s_l) + e(n + 2(s_i - m))$$

with $e \geqq 1$. We have

$$2(s_j - s_i) = c_0 \cdot 2m + \sum_{l=1}^{m-1} c_l \cdot 2s_l + d_0 \cdot n + \sum_{l=1}^{m-1} d_l(n + 2s_l) + (e-1)(n + 2(s_i - m)).$$

In view of $n \geqq 2c(H) - 1$ we obtain $d_0 = d_l = e - 1 = 0$, which implies that

$$2(s_j - s_i) = c_0 \cdot 2m + \sum_{l=1}^{m-1} c_l \cdot 2s_l.$$

So, we get

$$s_j = c_0 m + \sum_{l=1}^{m-1} c_l s_l + s_i.$$

By the definition of s_j we must have $s_j = s_k + s_i$ for some k, which contradicts the assumption ii). □

Since we have no j such that $s_j = s_{\max} + s_k$ for some k, by Lemma 2.2 we get the following:

Remark 2.3 *Let H be an m-semigroup and n an odd number $\geqq 2c(H) - 1$ with $n \neq 2m - 1$. We set*

$$\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0.$$

Then $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - 1$.

We can show the following, which means Theorem B in the case where $\tilde{H}$ is as in Remark 2.3.

Proposition 2.4 *Let H be an m-semigroup. Then the following are equivalent:*
i) *H is Weierstrass.*
ii) *There exists an odd number $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$ such that $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0$ is the double covering type.*
iii) *For any odd $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$ the semigroup $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0$ is the double covering type.*

Proof. It is trivial that iii) implies ii).

Assume that ii) holds. Then there exists a double covering $\pi : \tilde{C} \longrightarrow C$ of a curve with a ramification point $\tilde{P}$ such that $H(\tilde{P}) = \tilde{H}$. Hence, we get $H = d_2(\tilde{H}) = d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$, which implies that H is Weierstrass.

The main part of the proof is to show that i) implies iii). Assume that (C, P) is a pointed curve with $H(P) = H$. Let n be an odd number with $n \geqq 2c(H) - 1$ and $n \neq 2m - 1$. We set

$$D = \frac{n+1}{2}P - P_1$$

where P_1 is a point of C distinct from P. In view of the conditions on n we get $\deg(2D - P) \geqq 2g(H) + 1$. In fact, if $H = \langle m, m+1, \ldots, m+m-1 \rangle$, then $\deg(2D - P) = n - 2 \geqq 2m + 1 - 2 \geqq 2g(H) + 1$. Otherwise, we have

$$\deg(2D - P) = n - 2 \geqq 2c(H) - 3 \geqq 2(g(H) + 2) - 3 = 2g(H) + 1.$$

Thus, the complete linear system $|2D - P|$ is very ample, hence base-point free, which implies that

$$2D \sim P + Q_1 + \cdots + Q_{n-2}$$

where the points P, Q_1, ..., Q_{n-2} are distinct. We set $\mathcal{L} = \mathcal{O}_C(-D)$. Then there is an isomorphism $\mathcal{L}^{\otimes 2} \overset{\phi}{\cong} \mathcal{O}_C(-(P + Q_1 + \cdots + Q_{n-2})) \subset \mathcal{O}_C$. Hence, the direct sum $\mathcal{O}_C \oplus \mathcal{L}$ has an $\mathcal{O}_C$-algebra structure through ϕ. Let $\pi : \tilde{C} = \mathrm{Spec}\ (\mathcal{O}_C \oplus \mathcal{L}) \longrightarrow C$ be a canonical morphism. We note that the genus $g(\tilde{C})$ of $\tilde{C}$ is $2g(H) + \dfrac{n-1}{2} - 1$, because the branch divisor of π is $P + Q_1 + \cdots + Q_{n-2}$. Let $\tilde{P} \in \tilde{C}$ such that $\pi(\tilde{P}) = P$. By Proposition 2.1 in [9] we obtain

$$\begin{aligned} l((n-1)\tilde{P}) &= l(\frac{n-1}{2}P) + l(\frac{n-1}{2}P - D) \\ &= l(\frac{n-1}{2}P) + l(P_1 - P) = l(\frac{n-1}{2}P), \end{aligned}$$

because of $P_1 \neq P$. Here, we denote $\dim_k H^0(X, \mathcal{O}_X(D))$ by $l(D)$ for a divisor D on a curve X. Moreover, we have

$$l((n+1)\tilde{P}) = l(\frac{n+1}{2}P) + l(P_1) = l(\frac{n+1}{2}P) + 1.$$

By the assumption $n \geqq 2c(H) - 1$ we get $\dfrac{n+1}{2} \geqq c(H)$. Hence,

$$l(\frac{n+1}{2}P) = l(\frac{n-1}{2}P) + 1,$$

which implies that $n \in H(\tilde{P})$. Thus, it suffices to show that

$$n + 2(s_{\max} - m) \in H(\tilde{P})$$

because of $g(H(\tilde{P})) = g(\tilde{C})$ and Remark 2.3. We obtain

$$l((n + 2(s_{\max} - m) - 1)\tilde{P}) = l((s_{\max} - m + \frac{n-1}{2})P) + l((s_{\max} - m - 1)P + P_1)$$

and

$$l((n + 2(s_{\max} - m) + 1)\tilde{P}) = l((s_{\max} - m + \frac{n-1}{2} + 1)P) + l((s_{\max} - m)P + P_1).$$

On the other hand we have

$$l(K - (s_{\max} - m - 1)P) = l(K - (s_{\max} - m)P) + 1 = 1,$$

because $s_{\max} - m$ is the largest gap at P, where K is a canonical divisor on C. Hence there exists a unique effective divisor E on C such that

$$E \sim K - (s_{\max} - m - 1)P.$$

Let us take the point $P_1 \not< E$. Then we have

$$l(K - (s_{\max} - m - 1)P - P_1)) = 0,$$

which implies that

$$l((s_{\max} - m)P + P_1) = l((s_{\max} - m - 1)P + P_1) + 1.$$

Hence, we get $n + 2(s_{\max} - m) \in H(\tilde{P})$. □

3 Semigroups $\tilde{H}$ with symmetric $d_2(\tilde{H})$

In this section we will give a characterization of $\tilde{H}$ similar to Remark 2.1 in the case where $H = d_2(\tilde{H})$ is a symmetric m-semigroup, i.e., $2g(H) - 1 = s_{\max} - m$, with $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - 1$. It is well-known that a necessary and sufficient condition for H to be symmetric is the following:

$$\gamma \in \mathbb{Z}\backslash H \text{ if and only if } 2g(H) - 1 - \gamma \in H.$$

Lemma 3.1 *If H is a symmetric m-semigroup, then we have the following: if $i \in \{1, 2, \ldots, m-1\}$ satisfies the condition where there is no j such that $s_j = s_i + s_k$ for some k, then $s_i = s_{\max}$.*

Proof. Since H is symmetric, we get

$$2g(H) - 1 - (s_i - m) \in H \text{ for any } i.$$

Moreover, we have

$$s_{\max} = 2g(H) - 1 + m = 2g(H) - 1 - (s_i - m) + s_i.$$

Since $s_{\max}$ belongs to $S(H)$, for $s_i \neq s_{\max}$ there is some k such that

$$2g(H) - 1 - (s_i - m) = s_k.$$

Thus, we have the following: if $s_i \neq s_{\max}$, then $s_{\max} = s_i + s_k$ for some k. □

We want to prove the converse of Lemma 3.1 which we do in the following two lemmas.

Lemma 3.2 *Let H be an m-semigroup. Assume that we have $s_q = s_{\max}$ if $q \in \{1, \ldots, m-1\}$ has no j such that $s_j = s_q + s_k$ for some k. Then for any i with $s_i \neq s_{\max}$ we have $s_{\max} = s_i + s_k$ for some k.*

Proof. We set

$$S(H)\backslash\{m\} = \{s_{\max} = s^{(1)} > s^{(2)} > \cdots > s^{(m-1)}\}.$$

By the assumption there exists some $k^{(2)}$ such that $s_{j_2} = s^{(2)} + s_{k^{(2)}} > s^{(2)}$, which implies that $s_{j_2} = s_{\max}$. We prove the statement by induction on l with $s^{(l)}$. Assume that for any $2 \leqq \nu \leqq l$, we have $s_{\max} = s^{(\nu)} + s_{k^{(\nu)}}$. By the assumption $s_{j_{l+1}} = s^{(l+1)} + s_{k^{(l+1)}} > s^{(l+1)}$. Hence, we have $s^{(\mu)} = s^{(l+1)} + s_{k^{(l+1)}}$ for some μ with $1 \leqq \mu \leqq l$. If $s^{(\mu)} \neq s_{\max}$, then by the induction hypothesis we have

$$s_{\max} = s^{(\mu)} + s_{k^{(\mu)}} = s^{(l+1)} + s_{k^{(l+1)}} + s_{k^{(\mu)}}.$$

Hence, we must have $s_{\max} = s^{(l+1)} + s_p$ for some p. □

Lemma 3.3 *Let H be an m-semigroup such that for any $i \in \{1, \ldots, m-1\}$ with $s_i \neq s_{\max}$ we have $s_{\max} = s_i + s_k$ for some k. Then H is symmetric.*

Proof. Let $s_{\max} = ml + n$ with a positive integer $n \leqq m-1$. Now we have the following equalities:

$$\sum_{i=1}^{n-1} \left[\frac{s_i + s_{n-i}}{m}\right] + \sum_{i=1}^{m-n-1} \left(\left[\frac{s_{n+i} + s_{m-i}}{m}\right] - 1\right)$$

$$= \sum_{i=1}^{n-1} \left(\left[\frac{s_i}{m}\right] + \left[\frac{s_{n-i}}{m}\right]\right) + \sum_{i=1}^{m-n-1} \left(\left[\frac{s_{n+i}}{m}\right] + \left[\frac{s_{m-i}}{m}\right]\right)$$

$$= 2\sum_{i=1}^{m-1} \left[\frac{s_i}{m}\right] - 2\left[\frac{s_{\max}}{m}\right] = 2g(H) - 2\left[\frac{s_{\max}}{m}\right].$$

By the assumption we obtain

$$\sum_{i=1}^{n-1} \left[\frac{s_i + s_{n-i}}{m}\right] + \sum_{i=1}^{m-n-1} \left(\left[\frac{s_{n+i} + s_{m-i}}{m}\right] - 1\right)$$

$$= \sum_{i=1}^{n-1} \left[\frac{s_{\max}}{m}\right] + \sum_{i=1}^{m-n-1} \left(\left[\frac{s_{\max}}{m}\right] - 1\right) = (m-2)\left[\frac{s_{\max}}{m}\right] - m + n + 1.$$

Hence,

$$2g(H) = m\left[\frac{s_{\max}}{m}\right] - m + n + 1 = s_{\max} - (m-1) = c(H),$$

which implies that H is symmetric. □

Thus, by Lemmas 3.1, 3.2 and 3.3 we get another characterization of a symmetric semigroup as follows:

Lemma 3.4 *Let H be a numerical semigroup. The following are equivalent:*
i) *H is symmetric.*
ii) *For any i, if $s_i \neq s_{\max}$, then $s_i + s_k \in S(H)$ for some k.*
iii) *For any i, if $s_i \neq s_{\max}$, then $s_i + s_k = s_{\max}$ for some k.*

By Lemmas 2.2 and 3.4 we have the following:

Remark 3.5 *Let H be a non-symmetric m-semigroup and n an odd number $\geqq 2c(H) - 1$ with $n \neq 2m - 1$. Then there exists a numerical semigroup $\tilde{H}$ with $d_2(\tilde{H}) = H$ and $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h}$ is odd $\}$ such that*

$$g(\tilde{H}) = 2g(H) + \frac{n-1}{2} - 1 \text{ and } \tilde{H} \neq 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0.$$

By Lemma 2.2, Propositions 2.4 and Lemma 3.4 we gain a result similar to Remark 2.1 in the case where we discuss in this section.

Theorem 3.6 *Let H be a symmetric m-semigroup and $\tilde{H}$ a numerical semigroup with $d_2(\tilde{H}) = H$. We set $n = \min\{\tilde{h} \in \tilde{H} \mid \tilde{h}$ is odd $\}$. Assume that $n \geqq 2c(H) - 1$ with $n \neq 2m - 1$.*
Then the following are equivalent:
i) $g(\tilde{H}) = 2g(H) + \dfrac{n-1}{2} - 1$,
ii) $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0$.
In this case, if H is Weierstrass, $\tilde{H}$ is the double covering type.

4 A characterization of a semigroup of double covering type by a pair of points

In this section we give the proof of Theorem B. First, using the notation of $H(P,Q)$ we can put in another way the essential part of the proof of Proposition 2.4 as follows:

Remark 4.1 *Let H be a Weierstrass m-semigroup. Take a pointed curve (C,P) with $H(P) = H$. Let E be a unique effective divisor on C which is linearly equivalent to $K - (s_{\max} - m - 1)P$. If a point P_1, distinct from P, satisfies $P_1 \not\leq E$, then $(s_{\max} - m, 1) \in H(P, P_1)$.*

Proof. We note that $(n,l) \in H(P,Q)$ with positive integers n and l if and only if

$$l(nP + lQ) = l((n-1)P + lQ) + 1 = l(nP + (l-1)Q) + 1.$$

Since $s_{\max} - m$ is the last gap at P, we get

$$l((s_{\max} - m)P + P_1) = l((s_{\max} - m)P) + 1.$$

In view of $P_1 \not\leq E$ we have

$$l((s_{\max} - m)P + P_1) = l((s_{\max} - m - 1)P + P_1) + 1.$$

Hence, we obtain $(s_{\max} - m, 1) \in H(P, P_1)$. □

Here we prove Theorem B. In terms of Weierstrass semigroups of pairs of points the theorem gives a characterization of an m-semigroup H such that $\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_i - m))\mathbb{N}_0$ is the double covering type.

The proof of Theorem B

iii)$\Rightarrow$ii) We set $D = \dfrac{n+1}{2}P - P_1$. Using the method as in the proof of Proposition 2.4 we construct a pointed curve $(\tilde{C}, \tilde{P})$ such that $\tilde{C}$ is a double covering of C with a ramification point $\tilde{P}$. In view of $(s_i - m, 1) \in H(P, P_1)$ we have

$$l((s_i - m)P + P_1) = l((s_i - m - 1)P + P_1) + 1,$$

which, as seen in the proof of Proposition 2.4, is enough to conclude that $H(\tilde{P}) \ni n + 2(s_i - m)$. Hence, we get $\tilde{H} = H(\tilde{P})$, i.e., $\tilde{H}$ is the double covering type.
ii)$\Rightarrow$i) It is trivial.
i)$\Rightarrow$iii) Let $(\tilde{C}, \tilde{P})$ be a pointed curve with $H(\tilde{P}) = \tilde{H}$ and $\pi : \tilde{C} \longrightarrow C$ a double covering which is ramified at $\tilde{P}$. We set $P = \pi(\tilde{P})$ with $H(P) = H$. Then the double covering π can be constructed using a divisor $D = \dfrac{n+1}{2}P - P_1$ with $P_1 \neq P$ on C in the usual way as in the proof of Proposition 2.4, because $n \in \tilde{H}$ implies that

$$l(\frac{n+1}{2}P - D) = l(\frac{n-1}{2}P - D) + 1 = 1.$$

Since $n + 2(s_i - m) \in H(\tilde{P})$, we must have

$$l((s_i - m)P + P_1) = l((s_i - m - 1)P + P_1) + 1,$$

which implies that

$$l((s_i - m)P) + 1 \geqq l((s_i - m)P + P_1) \geqq l((s_i - m - 1)P) + 1.$$

The integer $s_i - m$ is a gap at P. Hence, using the above inequalities we get

$$l((s_i - m)P) + 1 = l((s_i - m)P + P_1),$$

which implies that $(s_i - m, 1) \in H(P, P_1)$. $\square$

Using Theorem B we give numerical semigroups $\tilde{H}$ of double covering type with $d_2(\tilde{H}) = \langle 4, 5, 6, 7 \rangle$.

Example 4. 1 Let $H = \langle 4, 5, 6, 7 \rangle$. Then

$$S(H) = \{m = 4, s_1 = 5, s_2 = 6, s_{\max} = s_3 = 7\}.$$

Let n be an odd number $\geqq 2c(H) - 1 = 7$ with $n \neq 2m - 1 = 7$, i.e., $n \geqq 9$.
i) By Proposition 2.4 we see that

$$\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_{\max} - m))\mathbb{N}_0 = \langle 8, 10, 12, 14, n, n+6 \rangle$$

is the double covering type.
ii) Using Theorem B and the method in [5] we can show that

$$\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_2 - m))\mathbb{N}_0 = \langle 8, 10, 12, 14, n, n+4 \rangle$$

is the double covering type. In fact, let C be a plane curve defined by

$$a(yz^3 - x^3z) + b(xy^3 + y^2z^2) = 0$$

where a and b are general constants. We set $P = (0 : 1 : 0)$ which is a point of C. Then the tangent line at P is $x = 0$, hence $H(P) = \langle 4, 5, 6, 7\rangle$. Moreover, we have $K \sim 2P + P_1 + Q$ where we set $P_1 = (0 : 0 : 1)$ and $Q = (0 : -a : b)$. Hence, we get $l(K - 2P - P_1) = 1$, which implies that $(2, 1) \in H(P, P_1)$. By Theorem B, $\tilde{H}$ is the double covering type.
iii) Let C be a hyperelliptic curve of genus 3 and P an ordinary point. Take a unique ordinary point $P_1 \in C$ with $P + P_1 \sim g^1_2$. Then we have $(1, 1) \in H(P, P_1)$. Hence, by Theorem B

$$\tilde{H} = 2H + n\mathbb{N}_0 + (n + 2(s_1 - m))\mathbb{N}_0 = \langle 8, 10, 12, 14, n, n + 2\rangle$$

is the double covering type.

5 Non-Weierstrass 8-semigroups and non-Weierstrass 12-semigroups

In this section we solve Problem A negatively and prove Main Theorem. First, we prove the following lemma:

Lemma 5.1 *Let $l \geqq 2$. The semigroup*

$$H = \langle 4, 6, 4l + 1, 4l + 3\rangle \quad (\textit{resp.}\ \langle 4, 6, 4l + 3, 4(l + 1) + 1\rangle)$$

cannot be attained by a pointed trigonal curve. In particular, if (C, P) is a pointed curve such that $H(P)$ is the above semigroup H, then for any point P_1 of C, distinct from P, we have $(2, 1) \notin H(P, P_1)$.

Proof. Let (C, P) be a pointed curve such that $H(P)$ is the semigroup H as in the statement. We will show that C is not a trigonal curve. Assume that C were trigonal. Then we have two morphisms $\varphi_{|4P|}$ and $\varphi_{g^1_3}$ from C to $\mathbb{P}^1$ determined by the linear systems $|4P|$ and g^1_3 respectively. Hence, we see that the genus $g(C)$ of C is less than or equal to 6. Since $g(H) = 2l + 1$ (resp. $2l + 2$), we may assume that $l = 2$. By Coppens' assertions on page 9 in [2] and page 11 in [3] and Kim's assertion on page 3 in [4] this is a contradiction. □

Using Lemma 5.1 and Theorem B we get the following:

Theorem 5.2 *Let $l \geqq 2$ and n an odd number $\geqq 8l - 1$ (resp. $8l + 3$). Then*

$$\tilde{H} = \langle 8, 12, 8l + 2, 8l + 6, n, n + 4\rangle \text{ (resp. } \langle 8, 12, 8l + 6, 8l + 10, n, n + 4\rangle\}$$

is not the double covering type.

Proof. Assume that $\tilde{H}$ were the double covering type. By Theorem B i)⇒ iii) there exists a pointed curve (C, P) with $H(P) = \langle 4, 6, 4l + 1, 4l + 3\rangle$ (resp. $\langle 4, 6, 4l + 3, 4l + 5\rangle$) such that $(2, 1) \in H(P, P_1)$. This contradicts Lemma 5.1. □

Hence, if we take sufficiently large n, Problem A has been solved negatively. In fact, we have the following counterexamples :

Example 5.1 Let $l \geqq 2$ and n an odd number with $n \geqq 16l + 19$ (resp. $16l + 27$). We set

$$\tilde{H} = \langle 8, 12, 8l + 2, 8l + 6, n, n + 4\rangle \text{ (resp. } \langle 8, 12, 8l + 6, 8l + 10, n, n + 4\rangle).$$

Then

$$d_2(\tilde{H}) = \langle 4, 6, 4l + 1, 4l + 3\rangle \quad \text{(resp. } \langle 4, 6, 4l + 3, 4(l + 1) + 1\rangle)$$

is Weierstrass by [6]. Moreover, we have

$$g(\tilde{H}) = 2g(d_2(\tilde{H})) + \frac{n-1}{2} - 1 \geqq 6g(d_2(\tilde{H})) + 4.$$

But by Theorem 5.2 $\tilde{H}$ is not the double covering type.

Theorem 5.3 *Let $l \geqq 2$ and n an odd number with $n \geqq 12l - 1$. Then*

$$\tilde{H} = \langle 12, 16, 20, 12l + 2, 12l + 6, 12l + 10, n, n + 4\rangle$$

is not the double covering.

Proof. By [2] , [3] and [4] the semigroup $d_2(\tilde{H}) = \langle 6, 8, 10, 6l + 1, 6l + 3, 6l + 5\rangle$ of genus $3l + 2$ cannot be attained by a pointed trigonal curve. By Theorem B i)⇒ iii) we get the desired result. □

By Torres' assertion on page 2 in [12], Example 5.1 and Theorem 5.3 we can prove Main Theorem.

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